# SPECIAL FUNCTIONS



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[*For Honours and Post-Graduate Students of Various Universities*]

*By*

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## PREFACE TO THE FIRST EDITION

The book entitled 'Special Functions' has been written to meet the requirements of the students of various Indian Universities on this subject.

Each topic in the book has been treated in an easy and lucid style without sacrificing any rigor. The language is simple and easily understandable. The book contains a large number of worked out examples. These example provide the students with model solutions.

No claim to originality can be made but the treatment of the subject is in our style. Suggestions for the improvement of the book will be gratefully acknowledged.

The authors also feel thankful to the publishers and printers for their full co-operation in bringing out the book in the present nice form.

—*The Authors*

## PREFACE TO THE SEVENTEENTH EDITION

In this edition the book has been thoroughly revised and enlarged. A large number of new examples have been added.

—*The Authors*

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# 1
# Spherical Harmonics

## § 1·1. Spherical Harmonics.

In dealing with the theory of potential, we commonly use the Laplace's equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0.$$

Any solution $V_n$ of this equation, which is a homogeneous polynomial (of degree $n$) in $x, y, z$ is called the **Solid Spherical Harmonics.**

Laplace's equation in polar or spherical coordinates is written as

$$\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}=0.$$

Let $V_n$ ; when expressed in polar coordinates take the form $V_n=r^n.U_n(\theta, \phi)$, where $U_n(\theta, \phi)$ is a function of $\theta$ and $\phi$ only.

Then $U_n(\theta, \phi)$ is called the **Surface Spherical Harmonics** of degree $n$.

Obviously when $r=1$, $\qquad V_n=U_n$.

Hence surface spherical harmonics is the value of the corresponding solid spherical harmonics on the surface of a unit sphere whose centre is the origin.

**Ex.** *Show that* $(z+ix\cos\alpha+iy\sin\alpha)^n$ *is a solid spherical harmonics of degree n.*

**Sol.** Let $V=(z+ix\cos\alpha+iy\sin\alpha)^n$

$$\therefore \quad \frac{\partial V}{\partial x}=n(z+ix\cos\alpha+iy\sin\alpha)^{n-1}\, i\cos\alpha$$

and
$$\frac{\partial^2 V}{\partial x^2}=n(n-1)(z+ix\cos\alpha+iy\sin\alpha)^{n-2}\, i^2\cos^2\alpha$$

$$=-n(n-1)(z+ix\cos\alpha+iy\sin\alpha)^{n-2}\cos^2\alpha.$$

Similarly, $\dfrac{\partial^2 V}{\partial y^2}=-n(n-1)(z+ix\cos\alpha+iy\sin\alpha)^{n-2}\sin^2\alpha$

and $\dfrac{\partial^2 V}{\partial z^2}=n(n-1)(z+ix\cos\alpha+iy\sin\alpha)^{n-2}$

Clearly $$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

*i.e.* $V$ satisfies the Laplace's equation and is homogeneous function of degree $n$ in $x, y, z$.

**§ 1·2. Kelvin's Theorem.**

*If $V_n$ is a solid spherical harmonics of degree $n$, then $\frac{V_n}{r^{2n+1}}$ is a solid spherical harmonics of degree $-(n+1)$.*

**Proof.** Let $$V=r^m V_n.$$

If $V=r^m V_n$ is a solid spherical harmonics, it must satisfy the Laplace's equation.

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0. \qquad \text{...(i)}$$

Now $\frac{\partial V}{\partial x}=r^m\frac{\partial V_n}{\partial x}+r^{m-1}\frac{\partial r}{\partial x}V_n.$ Since $\frac{\partial r}{\partial x}=\frac{x}{r}$

from $r^2=x^2+y^2+z^2$

$$=r^m\frac{\partial V_n}{\partial x}+mx\, r^{m-2}\, V_n$$

and $$\frac{\partial^2 V}{\partial x^2}=r^m\frac{\partial^2 V_n}{\partial x^2}+mr^{m-1}\frac{\partial r}{\partial x}\frac{\partial V_n}{\partial x}+mx\, r^{m-2}\frac{\partial V_n}{\partial x}$$
$$+m(m-2)\, xr^{m-3}\frac{\partial r}{\partial x}V_n+mr^{m-2}V_n$$

$$=r^m\frac{\partial^2 V_n}{\partial x^2}+2m\, xr^{m-2}\frac{\partial V_n}{\partial x}+m(m-2)\, x^2 r^{m-4}V_n+mr^{m-2}V_n$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2}=r^m\frac{\partial^2 V_n}{\partial y^2}+2m\, yr^{m-2}\frac{\partial V_n}{\partial y}+m(m-2)\, y^2 r^{m-4}V_n+mr^{m-2}V_n$$

and $$\frac{\partial^2 V}{\partial z^2}=r^m\frac{\partial^2 V_n}{\partial z^2}+2m\, zr^{m-2}\frac{\partial V_n}{\partial z}+m(m-1)\, z^2 r^{m-4}V_n+mr^{m-2}V_n.$$

Substituting the values in (i), we have

$$r^m\left(\frac{\partial^2 V_n}{\partial x^2}+\frac{\partial^2 V_n}{\partial y^2}+\frac{\partial^2 V_n}{\partial z^2}\right)+2mr^{m-2}\left(x\frac{\partial V_n}{\partial x}+y\frac{\partial V_n}{\partial y}+z\frac{\partial V_n}{\partial z}\right)$$
$$+m(m-2)\, r^{m-4}(x^2+y^2+z^2)V_n+3mr^{m-2}V_n=0$$

or $$r^m\, 0+2mr^{m-2}\, nV_n+m(m-2)\, r^{m-4}\, r^2 V_n+3mr^{m-2}V_n=0.$$

Since $\frac{\partial^2 V_n}{\partial x^2}+\frac{\partial^2 V_n}{\partial y^2}+\frac{\partial^2 V_n}{\partial z^2}=0$, $x\frac{\partial V_n}{\partial x}+y\frac{\partial V_n}{\partial y}+z\frac{\partial V_n}{\partial z}=nV_n$

as $V_n$ is solid spherical harmonics of degree $n$ and $V_n$ is also homogeneous function of degree $n$.

or $$m(2n+m-2+3)\, r^{m-2}V_n=0$$

or $$m(m+2n+1)\, r^{m-2}V_n=0.$$

$$\therefore \quad m=0 \quad \text{or} \quad m=-2n-1.$$

Thus $V=r^{-2n-1}V_n$ is a solid spherical harmonics of degree $-(n+1)$.

## § 1·3. Legendre's equation from Laplace's equation

Laplace's equation in spherical coordinates is

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0 \qquad \text{...(i)}$$

Putting $V=r^nU_n$, where $U_n$ is a function of $\theta$ and $\phi$ only

so that $\dfrac{\partial V}{\partial r}=nr^{n-1}U_n,\ \dfrac{\partial V}{\partial\theta}=r^n\dfrac{\partial U_n}{\partial\theta},\ \dfrac{\partial^2V}{\partial\phi^2}=r^n\dfrac{\partial^2U_n}{\partial\phi^2}.$

Substituting in (i), we have

$$\frac{\partial}{\partial r}(nr^{n+1}U_n)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(r^n\sin\theta\frac{\partial U_n}{\partial\theta}\right)+\frac{1}{\sin^2\theta}.r^n\frac{\partial^2U_n}{\partial\phi^2}=0$$

or $$n(n+1)r^nU_n+\frac{r^n}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial U_n}{\partial\theta}\right\}+\frac{r^n}{\sin^2\theta}\frac{\partial^2U_n}{\partial\phi^2}=0$$

or $$n(n+1)U_n+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U_n}{\partial\theta}\right)=0. \qquad \text{...(ii)}$$

Supposing $U_n$ to be independent of $\phi$.

Putting $\mu=\cos\theta$ and $U_n=y$

so that $$\frac{\partial U_n}{\partial\theta}=\frac{\partial y}{\partial\theta}=\frac{\partial y}{\partial\mu}\cdot\frac{\partial\mu}{\partial\theta}=-\sin\theta\frac{\partial y}{\partial\mu}.$$

Substituting in (ii), we have

$$n(n+1)y+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(-\sin^2\theta\frac{\partial y}{\partial\mu}\right)=0$$

or $$n(n+1)y+\frac{1}{\sin\theta}\frac{\partial}{\partial\mu}\left\{-(1-\mu^2)\frac{\partial y}{\partial\mu}\right\}\cdot\frac{\partial\mu}{\partial\theta}=0$$

or $$\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial y}{\partial\mu}\right\}+n(n+1)y=0.$$

which is Legendre's equation.

Solution $P_n(\mu)$ of Legendre's equation is the surface spheric harmonic of degree $n$, which is free from $\phi$.

## § 1·4. Bessel's Equation from Laplace's Equation

$$\frac{\partial^2V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2V}{\partial\theta^2}+\frac{\partial^2V}{\partial z^2}=0. \qquad \text{...(i)}$$

Now let $V=R\theta'z'$ where $R,\theta',z'$ are functions of $r,\theta$ and $z$ alone, respectively.

Substituting in (i), we have

$$\theta'z'\frac{d^2R}{dr^2}+\frac{1}{r}\theta'z'\frac{dR}{dr}+\frac{1}{r^2}Rz'\frac{d^2\theta'}{d\theta^2}+R\theta'\frac{d^2z'}{dz^2}=0.$$

or $$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\frac{1}{r^2}\frac{1}{\theta'}\frac{d^2\theta'}{d\theta^2}+\frac{1}{z'}\frac{d^2z'}{dz^2}=0. \qquad \text{...(ii)}$$

Since the first three terms are independent of $z$ therefore the fourth term must also be independent of $z$. Thus the fourth term must be equal to a constant

i.e. $$\frac{1}{z'}\frac{d^2z'}{dz^2}=C \text{ (constant)} \quad \text{or} \quad \frac{d^2z'}{dz^2}=Cz'. \qquad \text{...(iii)}$$

Similarly third term must be free from $\theta$ and therefore must be equal to a constant

i.e. $$\frac{1}{\theta'}\frac{d^2\theta'}{d\theta^2}=D \quad \text{or} \quad \frac{d^2\theta'}{d\theta^2}=D\theta'. \qquad \text{...(iv)}$$

$\therefore$ With the help of (iii) and (iv), equation (ii), becomes

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\frac{1}{r^2}D+C=0$$

or $$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+(D+Cr^2)R=0. \qquad \text{...(v)}$$

Putting $kr=v$ so that $\frac{dR}{dr}=\frac{dR}{dv}\cdot\frac{dv}{dr}=k\frac{dR}{dv}$

and $$\frac{d^2R}{dr^2}=k\frac{d^2R}{dv^2}\cdot\frac{dv}{dr}=k^2\frac{d^2R}{dv^2}$$

in (v). We have

$$k^2r^2\frac{d^2R}{dv^2}+kr\frac{dR}{dv}+\left(D+C\frac{v^2}{k^2}\right)R=0.$$

Putting $C=k^2$, $D=-n^2$

or $$v^2\frac{d^2R}{dv^2}+v\frac{dR}{dv}+(v^2-n^2)R=0.$$

Putting $R=y$ and $v=x$, we have

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

which is the Bessel's equation.

The solution of this equation is called the cylindrical function or Bessel's function of order $n$.

# 2

# Legendre's Equation

**§ 2·1. Legendre's Equation. (Def.).** **[Kanpur 84]**

The differential equation of the form

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

is called Legendre's differential equation (or Legendre's equation), where $n$ is a constant.

This equation can also be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n(n+1)y=0.$$

**§ 2·2. Solution of Legendre's Equation.**

The Legendre's equation is

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0. \qquad \text{...(i)}$$

It can be solved in series of ascending or descending powers of $x$. The solution in descending powers of $x$ is more important than the one in ascending powers.

Let us assume, $y=\sum_{r=0}^{\infty} a_r x^{k-r}$

$$\therefore \quad \frac{dy}{dx}=\sum_{r=0}^{\infty} a_r (k-r) x^{k-r-1}.$$

and

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r (k-r)(k-r-1) x^{k-r-2}.$$

Substituting in (i), we have

$$(1-x^2)\sum_{r=0}^{\infty} a_r (k-r)(k-r-1) x^{k-r-2}-2x\sum_{r=0}^{\infty} a_r (k-r) k^{k-r-1}$$

$$+n(n+1)\sum_{r=0}^{\infty} a_r x^{k-r}=0$$

or $$\sum_{r=0}^{\infty} a_r \left[(k-r)(k-r-1)\, x^{k-r-2} + \{n(n+1)-(k-r)(k-r-1)-2(k-1)\}\, x^{k-r}\right]=0$$

or $$\sum_{r=0}^{\infty} a_r \left[(k-r)(k-r-1)\, x^{k-r-2} + \{n(n+1)-(k-r)(k-r+1)\}\, x^{k-r}\right]=0$$

or $$\sum_{r=0}^{\infty} a_r \left[(k-r)(k-r-1)\, x^{k-r-2} + \{n^2-(k-r)^2 + n-(k-r)\}\, x^{k-r}\right]=0$$

or $$\sum_{r=0}^{\infty} a_r \left[(k-r)(k-r-1)\, x^{k-r-2} + (n-k+r)(n+k-r+1)\, x^{k-r}\right]=0. \quad \text{...(ii)}$$

Now (ii) being an identity, we can equate to zero the coefficients of various powers of $x$.

$\therefore$ Equating to zero the coefficient of highest power of $x$, *i.e.* of $x^k$, we have

$$a_0\,(n-k)\,(n+k+1)=0.$$

Now $a_0 \neq 0$, as it is the coefficient of the first term with which we start to write the series.

$$\left.\begin{aligned} \therefore \quad & k=n \\ \text{or} \quad & k=-(n+1) \end{aligned}\right\} \quad \text{...(iii)}$$

Equating to zero the coefficient of the next lower power of $x$ *i.e.* of $x^{k-1}$, we have

$$a_1\,(n-k+1)\,(n+k)=0$$

$\therefore$ $a_1=0$, since neither $(n-k+1)$ nor $(n+k)$ is zero by virtue of (iii).

Again equating to zero the coefficient of the general term *i.e.* of $x^{k-r}$, we have

$$a_{r-2}\,(k-r+2)\,(k-r+1)+(n-k+r)\,(n+k-r+1)\,a_r=0$$

$$\therefore \quad a_r=-\frac{(k-r+2)\,(k-r+1)}{(n-k+r)\,(n+k-r+1)}\,a_{r-2}. \quad \text{...(iv)}$$

Putting $r=3$, $a_3=-\dfrac{(k-1)\,(k-2)}{(n-k+3)\,(n+k-2)}\,a_1$

$=0.$ since $a_1=0$

$\therefore$ we have, $a_1=a_3=a_5=\ldots=0$ (each).

Now two cases arise :

**Case I.** When $k=n$,
from (iv), we have

$$a_r=-\frac{(n-r+2)\,(n-r+1)}{r.(2n-r+1)}\,a_{r-2}$$

Putting $r=2, 4,\ldots$ etc.

$$a_2=-\frac{n.(n-1)}{2.(2n-1)}.a_0$$

$$a_4=-\frac{(n-2)\,(n-3)}{4.(2n-3)}.a_2$$

$$=\frac{n.(n-1)\,(n-2)\,(n-3)}{2.4.(2n-1)\,(2n-3)}.a_0$$

etc.

$$\therefore\quad y=a_0x^n+a_2x^{n-2}+a_4x^{n-4}+\ldots$$

or
$$y=a_0\left[x^n-\frac{n\,(n+1)}{2.(2n-1)}.x^{n-2}+\frac{n\,(n-1)\,(n-2)\,(n-3)}{2.4(2n-1).(2n-3)}\,x^{n-4}\ldots\right] \quad \ldots(\text{v})$$

which is one solution of Legendre's equation.

**Case II.** When $k=-(n+1)$

$$a_r=\frac{(n+r-1)\,(n+r)}{r\,(2n+r+1)}\,a_{r-2}.$$

Putting $r=2, 4, \ldots$etc.

$$a_2=\frac{(n+1)\,(n+2)}{2.(2n+3)}\,a_0$$

$$a_4=\frac{(n+3)\,(n+4)}{4.(2n+5)}\,a_2$$

$$=\frac{(n+1)\,(n+2)\,(n+3)\,(n+4)}{2.4.(2n+3)\,(2n+5)}\,a_0$$

etc.

$$\therefore\quad y=\sum_{r=0}^{\infty}a_r\,x^{-n-1-r}$$

$$=a_0\,x^{-n-1}+a_2x^{-n-3}+a_4x^{-n-5}+\ldots$$

$$=a_0\left[x^{-n-1}+\frac{(n+1)\,(n+2)}{2.(2n+3)}\,x^{-n-3}\right.$$

$$\left.+\frac{(n+1)\,(n+2)\,(n+3)\,(n+4)}{2.4.(2n+3)\,(2n+5)}\,x^{-n-5}+\ldots\right] \quad \ldots(\text{vi})$$

which is other solution of Legendre's equation.

### § 2·3. Definition of $P_n$ (x) and $Q_n$ (x). [Kanpur 83]

The solution of Legendre's equation is called Legendre's function.

When $n$ is a positive integer and

$a_0=\dfrac{1.3.5\ldots(2n-1)}{n\,!}$, the solution (v) of § 2·2 is denoted by $P_n(x)$ and is called **Legendre's function of the first kind.**

$$\therefore \quad P_n(x)=\frac{1.3.5\ldots(2n-1)}{n\,!}\left[x^n-\frac{n\,(n-1)}{2.(2n-1)}x^{n-2}\right.$$
$$\left.+\frac{n\,(n-1)\,(n-2)\,(n-3)}{2.4.(2n-1)\,(2n-3)}x^{n-4}\ldots\right]$$

$P_n\,(x)$ is a terminating series and gives what are called **Legendre's Polynomials** for different values of $n$.

We can write

$$P_n(x)=\sum_{r=0}^{(n/2)}(-1)^r\frac{(2n-2r)\,!}{2^n\,r\,!\,(n-2r)\,!\,(n-r)\,!}x^{n-2r}. \quad \text{[Kanpur 81]}$$

where $\left(\dfrac{n}{2}\right)=\begin{cases} n/2 \text{ if } n \text{ is even} \\ (n-1)/2 \text{ if } n \text{ is odd.}\end{cases}$

Again when $n$ is a positive integer and

$$a_0=\frac{n\,!}{1.3.5\,..(2n+1)},$$

the solution (vi) of § 2·2 is denoted by $Q_n\,(x)$ and is called the **Legendre's function of the second kind.**

$$\therefore \quad Q_n(x)=\frac{n\,!}{1.3\ldots(2n+1)}\left[x^{-n-1}+\frac{(n+1)\,(n+2)}{2\,(2n+3)}x^{-n-3}\right.$$
$$\left.+\frac{(n+1)\,(n+2)\,(n+3)\,(n+4)}{2.4.(2n+3)\,(2n+5)}x^{-n-5}+\ldots\right]$$

$Q_n\,(x)$ is an infinite or non-terminating series, as $n$, is positive.

**§ 2·4. General solution of Legendre's equation.** [Agra 84]

**The most general solution of the Legendre's equation is**

$$y=AP_n(x)+BQ_n(x),$$

where $A$ and $B$ are arbitrary constants.

**§ 2·5. To show that $P_n(x)$ is the coefficient of $h^n$ in the expansion in ascending powers of $(1-2xh+h^2)^{-1/2}$.**

[Raj. 79, 83; Agra 79, 83, 86; Meerut 79 (S), 81, 85, 90; Rohilkhand 82; Jodhpur 80, 83; Kanpur 84, 86]

We have

$$(1-2xh+h^2)^{-1/2}=\{1-h\,(2x-h)\}^{-1/2}$$
$$=1+\tfrac{1}{2}h\,(2x-h)+\frac{1.3}{2.4}h^2\,(2x-h)^2+\ldots$$
$$+\frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)}h^{n-1}\,(2x-h)^{n-1}$$
$$+\frac{1.3\ldots(2n-1)}{2.4\ldots(2n)}h^n\,(2x-h)^n+\ldots$$

$\therefore$ Coefficient of $h^n$

$$=\frac{1.3...(2n-1)}{2.4...2n}.(2x)^n+\frac{1.3...(2n-3)}{2.4...(2n-2)}\ {}^{n-1}C_1(2x)^{n-2}$$

$$+\frac{1.3...(2n-5)}{2.4...(2n-4)}\ {}^{n-2}C_2(2x)^{n-4}+...$$

$$=\frac{1.3...(2n-1)}{n\,!}\Big[x^n-\frac{2n}{2n-1}.(n-1).\frac{x^{n-2}}{2^2}$$

$$+\frac{2n\,(2n-2)}{(2n-1)\,(2n-3)}.\frac{(n-2)\,(n-3)}{2\,!}.\frac{x^{n-4}}{2^4}-...\Big]$$

$$=\frac{1.3...(2n-1)}{n\,!}\Big[x^n-\frac{n\,(n-1)}{2.(2n-1)}\,x^{n-2}$$

$$+\frac{n.(n-1)\,(n-2)\,(n-3)}{2.4\,(2n-1)\,(2n-2)}.x^{n-4}...\Big]$$

$=P_n(x)$.

Thus we can say that

$$\sum_{n=0}^{\infty} h^n\,P_n(x)=(1-2xh+h^2)^{-1/2}. \qquad \text{[Raj. 79, 83 ; Agra 86]}$$

**Note.** $(1-2xh+h^2)^{-1/2}$ is called the generating function of the Legendre polynomials.

**§ 2·6. Laplace's Definite integral for $P_n(x)$.**

**(1) Laplace's First Integral for $P_n(x)$ :** *When n is a positive integer,*

$$P_n(x)=\frac{1}{\pi}\int_0^{\pi}[x\pm\sqrt{(x^2-1)}\cos\phi]^n\,d\phi.$$

[Meerut 79 ; B.H.U. 72 ; Agra 81, 86; Kanpur 80, 85, 87 ; Raj. 85; Rohilkhand 80, 85; Jodhpur 83; Jiwaji 82]

**Proof.** From integral oalculus, we have

$$\int_0^{\pi}\frac{d\phi}{a\pm b\cos\phi}=\frac{\pi}{\sqrt{(a^2-b^2)}},\ \text{where } a^2>b^2.$$

Putting $a=1-hx$ and $b=h\sqrt{(x^2-1)}$

so that $a^2-b^2=(1-hx)^2-h^2\,(x^2-1)$

$$=1-2xh+h^2.$$

We have

$$\pi\,(1-2xh+h^2)^{-1/2}=\int_0^{\pi}[1-hx\pm h\sqrt{(x^2-1)}\cos\phi]^{-1}\,d\phi$$

$$=\int_0^{\pi}[1-h\,\{x\mp\sqrt{(x^2-1)}\cos\phi)\}]^{-1}\,d\phi$$

$$=\int_0^{\pi}[1-ht]^{-1}\,d\phi\ \text{where } t=x\mp\sqrt{(x^2-1)}\cos\phi$$

or $\pi \sum_{n=0}^{\infty} h^n P_n(x) = \int_0^{\pi} (1+ht+h^2t^2+\ldots+h^nt^n+\ldots)\, d\phi$

Equating the coefficients of $h^n$, we have

$$\pi P_n(x) = \int_0^{\pi} t^n \, d\phi = \int_0^{\pi} [x \mp \sqrt{(x^2-1)} \cos \phi]^n \, d\phi.$$

$$\therefore \quad P_n(x) = \frac{1}{\pi} \int_0^{\pi} [x \pm \sqrt{(x^2-1)} \cos \phi]^n \, d\phi.$$

**Note.** Putting $x = \cos \theta$, we have

$$P_n(\cos \theta) = \frac{1}{\pi} \int_0^{\pi} (\cos \theta + i \sin \theta \cos \phi)^n \, d\phi. \quad \textbf{[Kanpur 83]}$$

**(II) Laplace's Second Integral for $P_n(x)$ :** *When n is a positive integer.*

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \frac{d\phi}{[x \pm \sqrt{(x^2-1)} \cos \phi]^{n+1}}.$$

**[Agra 84; Kanpur 85; Jodhpur 82, 85]**

**Proof.** From integral calculus, we have

$$\int_0^{\pi} \frac{d\phi}{a \pm b \cos \phi} = \frac{\pi}{\sqrt{(a^2-b^2)}}, \text{ where } a^2 > b^2.$$

Putting $\quad a = xh - 1 \quad$ and $\quad b = h\sqrt{(x^2-1)}$

so that $\quad a^2 - b^2 = 1 - 2xh + h^2.$

We have

$$\pi (1-2xh+h^2)^{-1/2} = \int_0^{\pi} [-1 + xh \pm h\sqrt{(x^2-1)} \cos \phi]^{-1} \, d\phi$$

or $$\frac{\pi}{h}\left[1 - 2x\frac{1}{h} + \frac{1}{h^2}\right]^{-1/2} = \int_0^{\pi} [h\{x \pm \sqrt{(x^2-1)} \cos \phi\} - 1]^{-1} \, d\phi$$

or $$\frac{\pi}{h} \sum_{n=0}^{\infty} \frac{1}{h^n} P_n(x) = \int_0^{\pi} (t-1)^{-1} \, d\phi \text{ where } t = h\{x \pm \sqrt{(x^2-1)} \cos \phi\}$$

$$= \int_0^{\pi} \frac{1}{t}\left[1 - \frac{1}{t}\right]^{-1} d\phi$$

$$= \int_0^{\pi} \frac{1}{t}\left[1 + \frac{1}{t} + \frac{1}{t^2} + \ldots + \frac{1}{t^n} + \ldots\right] d\phi$$

$$= \int_0^{\pi} \left[\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \ldots + \frac{1}{t^{n+1}} + \ldots\right] d\phi$$

$$= \int_0^{\pi} \sum_{n=0}^{\infty} \frac{1}{t^{n+1}} \, d\phi$$

$$= \sum_{n=0}^{\infty} \int_0^{\pi} \frac{1}{h^{n+1} \{x \pm \sqrt{(x^2-1)} \cos \phi\}^{n+1}}$$

$\therefore$ Equating the coefficient of $\frac{1}{h^{n+1}}$, we have

$$\pi P_n(x)=\int_0^\pi \frac{d\phi}{\{x\pm\sqrt{(x^2-1)}\cos\phi\}^{n+1}}$$

or
$$P_n(x)=\frac{1}{\pi}\int_0^\pi \frac{d\phi}{\{x\pm\sqrt{(x^2-1)}\cos\phi\}^{n+1}}.$$

**Note.** Replacing $n$ by $-(n+1)$ in Laplace's second integral we have

$$P_{-(n+1)}(x)=\frac{1}{\pi}\int_0^\pi \{x\pm\sqrt{(x^2-1)}\cos\phi\}^n\,d\phi=P_n(x)$$

from Laplace's first integral

Hence $P_{-(n+1)}=P_n$. [Kanpur 85]

## § 2·7. Orthogonal Properties of Legendre's Polynomials.

(i) $\int_{-1}^{+1} P_m(x)\,P_n(x)\,dx=0$ if $m\neq n$.

[I.A.S. 81; Agra 78, 80, 82; Meerut 78, 80, 81 (P), 83, 86, 86 (R), 89; Rohilkhand 82, 84, 88; Gorakhpur 84]

(ii) $\int_{-1}^{+1} [P_n(x)]^2\,dx=\frac{2}{2n+1}$ if $m=n$.

[I.A.S. 81; Agra 78, 80, 82; Meerut 78, 81 (P), 82, 83, 84 (P), 86, 86 (R), 89; Rohilkhand 83, 85; Jodhpur 81, 83, 86; Kanpur 80, 86, 88]

**Proof.** Legendre's equation may be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n(n+1)y=0.$$

$$\therefore \quad \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}+n(n+1)P_n=0. \qquad \text{...(i)}$$

and
$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}+m(m+1)P_m=0. \qquad \text{...(ii)}$$

Multiplying (i) by $P_m$ and (ii) by $P_n$ and then subtracting, we have

$$P_m\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}-P_n\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}+\{n(n+1)-m(m+1)\}P_nP_m=0.$$

Integrating between the limits $-1$ to 1, we have

$$\int_{-1}^{+1} P_m\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx-\int_{-1}^{1} P_n\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}dx+\{n(n+1)-m(m+1)\}\int_{-1}^{1} P_mP_n\,dx=0$$

Integrating by parts

$$\left[P_m(1-x^2)\frac{dP_n}{dx}\right]_{-1}^{+1}-\int_{-1}^{+1}\frac{dP_m}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx$$

$$-\left[P_n(1-x^2)\frac{d}{dx}P_m\right]_{-1}^{+1}+\int_{-1}^{+1}\frac{dP_n}{dx}\left\{(1-x^2)\frac{d}{dx}P_m\right\}dx$$

$$+[n(n+1)-m(m+1)]\int_{-1}^{+1}P_mP_n\,dx=0.$$

$$\therefore\quad \{n(n+1)-m(m+1)\}\int_{-1}^{+1}P_mP_n\,dx=0.$$

Hence $\int_{-1}^{+1}P_m(x)\,P_n(x)\,dx=0$ since $m\neq n$.

(ii) We have

$$(1-2xh+h^2)^{-1/2}=\sum_{n=0}^{\infty}h^nP_n(x)$$

Squaring both sides, we have

$$(1-2xh+h^2)^{-1}=\sum_{n=0}^{\infty}h^{2n}\{P_n(x)\}^2+2\sum_{\substack{m=0\\n=0\\n\neq m}}^{\infty}h^{m+n}P_m(x)\,P_n(x)$$

Integrating between limits $-1$ to $+1$, we have

$$\sum_{n=0}^{\infty}\int_{-1}^{+1}h^{2n}[P_n(x)]^2\,dx+2\sum_{n=0}^{\infty}\int_{-1}^{+1}h^{m+n}P_m(x)\,P_n(x)\,dx$$

$$=\int_{-1}^{+1}\frac{dx}{(1-2xh+h^2)}$$

or $$\sum_{n=0}^{\infty}\int_{-1}^{+1}h^{2n}[P_n(x)]^2\,dx=\int_{-1}^{+1}\frac{dx}{(1-2xh+h^2)}$$

since other integral of the L.H.S. is zero by (1) as $m\neq n$.

$$=-\frac{1}{2h}\left\{\log(1-2xh+h^2)\right\}_{-1}^{+1}$$

$$=-\frac{1}{2h}\{\log(1-h)^2-\log(1+h)^2\}$$

$$=\frac{1}{2h}\left[\log\left\{\frac{1+h}{1-h}\right\}^2\right]=\frac{1}{h}\log\left\{\frac{1+h}{1-h}\right\}$$

$$=\frac{2}{h}\left\{h+\frac{h^3}{3}+\frac{h^5}{5}+\dots\right\}$$

$$=2\left\{1+\frac{h^2}{3}+\frac{h^4}{5}+\dots+\frac{h^{2n}}{2n+1}+\dots\right\}=\sum_{n=0}^{\infty}\frac{2h^{2n}}{2n+1}.$$

Equating the coefficients of $h^{2n}$, we have

$$\int_{-1}^{+1} [P_n(x)]^2 \, dx = \frac{2}{2n+1}.$$

## § 2·8. Recurrence formulae.

**(1) $(2n+1)\, xP_n = (n+1)\; P_{n+1} + nP_{n-1}$.**

**[Raj. 82; Kanpur 80, 81, 83; I.A.S. 79; Agra 83; Jodhpur 80, 81, 83; Meerut 80, 83 (P), 87; Rohilkhand 88; Gorakhpur 80, 84]**

**Proof.** We have

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n \, P_n(x)$$

Differentiating both sides w.r.t. '$h$' we have

$$-\tfrac{1}{2}\,(1-2xh+h^2)^{3/2}\,(-2x+2h) = \sum_{n=0}^{\infty} nh^{n-1} \; P_n(x)$$

or $$(x-h)\,(1-2xh+h^2)^{-1/2} = (1-2xh+h^2) \sum_{n=0}^{\infty} nh^{n-1} \; P_n(x)$$

or $$(x-h)\, \Sigma\, h^n P_n(x) = (1-2xh+h^2) \sum_{n=0}^{\infty} nh^{n-1} \; P_n(x)$$

or $$(x-h)\,[P_0(x)+hP_1(x)+\ldots+h^{n-1}\,P_{n-1}\,(x)+h^nP_n(x)+\ldots]$$
$$=(1-2xh+h^2)\,[P_1(x)+2hP_2(x)+\ldots+(n-1)\,h^{n-2}\,P_{n-1}\,(x)$$
$$+nh^{n-1}\,P_n(x)+(n+1)\,h^nP_{n+1}\,(x)+\ldots]$$

Equating the coefficients of $h^n$ from two sides, we have

$$xP_n(x)-P_{n-1}\,(x)=(n+1)\;P_{n+1}\,(x)-2xnP_n(x)+(n-1)\;P_{n-1}\,(x)$$

or $$(2n+1)\;xP_n(x)=(n+1)\;P_{n+1}\;(x)+nP_{n-1}\,(x).$$

**In short $(2n+1)\; xP_n = (n+1)\; P_{n+1} + nP_{n-1}$.**

**Note.** Equating the coefficients of $h^{n-1}$, from the two sides in (i), we get

$$xP_{n-1}(x)-P_{n-2}(x)=nP_n(x)-2x\,(n-1)\;P_{n-1}(x)+(n-2)\;P_{n-2}(x).$$

or **$nP_n = (2n-1)\; xP_{n-1} - (n-1)\; P_{n-2}$.**

**[Agra 89; Rohilkhand 83, 85; Raj. 84; Kanpur 86; Jodhpur 86]**

**(ii) $nP_n = xP'_n - P'_{n-1}$,** *where dashes denote differential w.r.t. '$x$'.*

**[I.A.S. 79; Raj. 81, 86; Meerut 81, 83(P), 88; Rohilkhand 80; Agra 79, 82, 85; Jiwaji 82; Kanpur 80, 84, 86]**

**Proof.** We have

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \qquad \ldots(\text{i})$$

Differentiating (i) w.r.t. '$h$', we have

$$(x-h)(1-2xh+h^2)^{-3/2}=\sum_{n=0}^{\infty} nh^{n-1} P_n(x). \qquad \text{...(ii)}$$

Again differentiating (1), w.r.t. '$x$', we have

$$h(1-2xh+h^2)^{-3/2}=\sum_{n=0}^{\infty} h^n P'_n(x)$$

or $$h(x-h)(1-2xh+h^2)^{-3/2}=(x-h)\sum_{n=0}^{\infty} h^n P'_n(x). \qquad \text{...(iii)}$$

From (ii) and (iii), we have

$$h\sum_{n=0}^{\infty} nh^{n-1} P_n(x)=(x-h)\sum_{n=0}^{\infty} h^n P'_n(x)$$

or $$h[h^0 P_1(x)+2hP_2(x)+\dots+nh^{n-1} P_n(x)+\dots]$$
$$=(x-h)[P'_0(x)+hP'_1(x)+\dots+h^{n-1} P'_{n-1}(x)+h^n P'_n(x)+\dots]$$

Equating the coefficient of $h^n$ on both sides, we have

$$nP_n(x)=xP'_n(x)-P'_{n-1}(x).$$

In short $\mathbf{nP_n=xP'_n-P'_{n-1}}$.

**(III)** $\mathbf{(2n+1)\,P_n=P'_{n+1}-P'_{n-1}}$.

**[Raj. 84; Kanpur 83; Agra 83, 85; Meerut 83(P), Rohilkhand 84]**

**Proof.** From reccurrence formula I, we have

$$(2n+1)\,xP_n=(n+1)\,P_{n+1}+nP_{n-1}.$$

Differentiating w.r.t. $x$, we have

$$(2n+1)\,xP'_n+(2n+1)\,P_n=(n+1)\,P'_{n+1}+nP'_{n-1}. \qquad \text{...(i)}$$

From reccurrence formula II, we have

$$xP'_n=nP_n+P'_{n-1}. \qquad \text{...(ii)}$$

Eliminating $xP'_n$ from (i) and (ii), we have

$$(2n+1)(nP_n+P'_{n-1})+(2n+1)\,P_n=(n+1)\,P'_{n+1}+nP'_{n-1}$$

or $$(2n+1)(n+1)\,P_n=(n+1)\,P'_{n+1}+nP'_{n-1}-(2n+1)\,P'_{n-1}$$

or $$(2n+1)(n+1)\,P_n=(n+1)\,P'_{n+1}-(n+1)\,P'_{n-1}$$

$$\therefore\quad \mathbf{(2n+1)\,P_n=P'_{n+1}-P'_{n-1}}.$$

**(IV)** $\mathbf{(n+1)\,P_n=(P'_{n+1}-xP'_n)}$. **[Meerut 84]**

**Proof** Writing reccurrence formula II and III, we have

$$nP_n=xP'_n-P'_{n-1} \qquad \text{...(i)}$$

and $$(2n+1)\,P_n=P'_{n+1}-P'_{n-1}. \qquad \text{...(ii)}$$

Substracting (i) from (ii), we have

$$\mathbf{(n+1)\,P_n=P'_{n+1}-xP'_n}.$$

**(V)** $(1-x^2)\,P'_n=n\,(P_{n-1}-xP_n)$.

**[Meerut 86 (R); Agra 84, 86, 88; Kanpur 84; Rohilkhand 84]**

**Proof.** Replacing $n$ by $(n-1)$ in reccurrence formula IV, we have

$$nP_{n-1}=P'_n-xP'_{n-1}. \qquad \ldots\text{(i)}$$

Writting II reccurrence formula, we have

$$nP_n=xP'_n-P'_{n-1} \qquad \ldots\text{(ii)}$$

multiplying by $x$ and then substracting from (i), we have

$$n\,(P_{n-1}-xP_n)=(1-x^2)P'_n,$$

*i.e.* $$(1-x^2)\,P'_n=n\,(P_{n-1}-xP_n).$$

**Aliter.** From Laplace's first integral, we have

$$P_n(x)=\frac{1}{\pi}\int_0^{\pi}\{x+\sqrt{(x^2-1)}\cos\phi\}^n\,d\phi.$$

Replacing $n$ by $(n-1)$, we have

$$P_{n-1}(x)=\frac{1}{\pi}\int_0^{\pi}\{x+\sqrt{(x^2-1)}\cos\phi\}^{n-1}\,d\phi$$

$$\therefore \quad P_{n-1}-xP_n=\frac{1}{\pi}\int_0^{\pi}[\{x+\sqrt{(x^2-1)}\cos\phi\}^{n-1}$$
$$-x\,\{x+\sqrt{(x^2-1)}\cos\phi\}^n]\,d\phi$$

$$=\frac{1}{\pi}\int_0^{\pi}\{x+\sqrt{(x^2-1)}\cos\phi\}^{n-1}$$
$$[1-x\,\{x+\sqrt{(x^2-1)}\cos\phi\}]\,d\phi$$

$$=\frac{1}{\pi}\int_0^{\pi}\{x+\sqrt{(x^2-1)}\cos\phi\}^{n-1}$$
$$[(1-x^2)-x\sqrt{(x^2-1)}\cos\phi]\,d\phi$$

$$=-\frac{(x^2-1)}{\pi}\int_0^{\pi}\{x+\sqrt{(x^2-1)}\cos\phi\}^{n-1}$$
$$\left\{1+\frac{x}{\sqrt{(x^2-1)}}\cos\phi\right\}d\phi$$

$$=-\frac{(x^2-1)}{\pi}\int_0^{\pi}\Big[\{x+\sqrt{(x^2-1)}\cos\phi\}^{n-1}$$
$$\times\frac{d}{dx}\{x+\sqrt{(x^2-1)}\cos\phi\Big]\,d\phi$$

$$=-\frac{(x^2-1)}{\pi}\int_0^{\pi}\left\{\frac{1}{n}\frac{d}{dx}.\{x+\sqrt{(x^2-1)}\cos\phi\}^n\right\}dx$$

$$=\frac{(1-x^2)}{\pi n}\frac{d}{dx}\int_0^{\pi}\{x+\sqrt{(x^2-1)}\cos\phi\}^n\,d\phi$$

$$=\frac{(1-x^2)}{\pi n}\frac{d}{dx}\{\pi\,P_n(x)\}$$

[from Laplace's first integral]

$$=\frac{(1-x^2)}{n}.P'_n(x).$$

Hence $(1-x^2)\,P'_n=n\,(P_{n-1}-xP_n)$

(VI) $(1-x^2)\,P'_n=(n+1)\,(xP_n-P_{n+1})$.

**Proof.** Writing reccurrence formula I, we have

$$(2n+1)\,xP_n=(n+1)\,P_{n+1}+nP_{n-1} \qquad \text{...(i)}$$

which may be written as

$$(n+1)\,xP_n+nxP_n=(n+1)\,P_{n+1}+nP_{n-1}$$

or

$$(n+1)\,(xP_n-P_{n+1})=n\,(P_{n-1}-xP_n).$$

Writing reccurrence formula V, we have

$$(1-x^2)\,P'_n=n\,(P_{n-1}-xP_n). \qquad \text{...(ii)}$$

From (i) and (ii), we have

$$(1-x^2)\,P'_n=(n+1)\,(xP_n-P_{n+1}).$$

**Aliter.** From Laplaces second integral, we have

$$P_n(x)=\frac{1}{\pi}\int_0^\pi \frac{d\phi}{[x+\sqrt{(x^2-1)}\cos\phi]^{n+1}}$$

$$\therefore\quad xP_n-P_{n+1}=\frac{x}{\pi}\int_0^\pi \frac{d\phi}{[x+\sqrt{(x^2-1)}\cos\phi]^{n+1}}-\frac{1}{\pi}\int_0^\pi \frac{d\phi}{[x+\sqrt{(x^2-1)}\cos\phi]^{n+2}}$$

$$=\frac{1}{\pi}\int_0^\pi [\{x+\sqrt{(x^2-1)}\cos\phi\}]^{-n-2}\,[x\,\{x+\sqrt{(x^2-1)}\cos\phi\}-1)]\,d\phi$$

$$=\frac{1}{\pi}\int_0^\pi [\{x+\sqrt{(x^2-1)}\cos\phi\}^{-n-2}\,\{(x^2-1)+x\sqrt{(x^2-1)}\cos\phi\}]\,d\phi$$

$$=\frac{(x^2-1)}{\pi}\int_0^\pi [\{x+\sqrt{(x^2-1)}\cos\phi\}^{-n-2}\,\{1+\frac{x}{\sqrt{(x^2-1)}}\cos\phi\}]\,d\phi$$

$$=\frac{(x^2-1)}{\pi}\int_0^\pi [\{x+\sqrt{(x^2-1)}\cos\phi)^{-n-2}\,\frac{d}{dx}\,[\{x+\sqrt{(x^2-1)}\cos\phi\}]\,d\phi$$

$$=-\frac{(x^2-1)}{\pi}\int_0^\pi \left[\frac{1}{n+1}\frac{d}{dx}\,\{x+\sqrt{(x^2-1)}\cos\phi\}\}^{-n-1}\right]d\phi$$

$$=\frac{(1-x^2)}{\pi\,(n+1)}\frac{d}{dx}\int_0^\pi \{x+\sqrt{(x^2-1)}\cos\phi)^{-n-1}\,d\phi$$

$$=\frac{(1-x^2\}}{\pi\,(n+1)}\frac{d}{dx}\,\{\pi P_n(x)\}$$

$$=\frac{(1-x^2)\,P'_n}{n+1}.$$

Hence $(1-x^2)\,P'_n=(n+1)\,(xP_n-P_{n+1})$.

**§ 2·9. Beltrami's results :** *To prove that*

$$(2n+1)\,(x^2-1)\,P'_n=n\,(n+1)\,(P_{n+1}-P_{n-1}),$$

**[Kanpur 87]**

**Proof.** From recurrence formula V and VI, we have

$$(1-x^2)\,P_n'=n\,(P_{n-1}-xP_n) \quad ...(i)$$

and $$(1-x^2)\,P'_n=(n+1)\,(xP_n-P_{n+1}). \quad ...(ii)$$

Substituting for $xP_n$ from (i) in (ii), we have

$$(1-x^2)\,P'_n=(n+1)\left[P_{n-1}-\frac{(1-x^2)}{n}\,P'_n-P_{n+1}\right]$$

or $$(1-x^2)\left\{1+\frac{(n+1)}{n}\right\}P'_n=(n+1)\,(P_{n-1}-P_{n+1})$$

or $$-(x^2-1)\,(2n+1)\,P'_n=n\,(n+1)\,(P_{n-1}-P_{n+1})$$

or $$(2n+1)\,(x^2-1)\,P'_n=n\,(n+1)\,(P_{n+1}-P_{n-1})$$

This result is known as **Beltrami's results.**

**§ 2·10. Christoffel's Expansion :** *To prove that*

$$P'_n=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+(2n-9)\,P_{n-5}+\cdots$$

*the last term of the series being $3P_1$ or $P_0$ according as n is, even or odd.* **[Kanpur 87; Agra 79; Rohilkhand 76; Raj 85]**

**Proof.** From Recurrence formula III, we have

$$P'_{n+1}=(2n+1)\,P_n+P'_{n-1} \quad ...(A)$$

Replacing $n$ by $(n-1)$, we have

$$P'_n=(2n-1)\,P_{n-1}+P'_{n-2}. \quad ...(i)$$

Replacing $n$ by $(n-2)$, $(n-4)$,...in (i), we have

$$P'_{n-2}=(2n-5)\,P_{n-3}+P'_{n-4} \quad ...(ii)$$

$$P'_{n-4}=(2n-9)\,P_{n-5}+P'_{n-6} \quad ...(iii)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$P'_2=3P_1+P_0'$$

when $n$ is even

Adding (i), (ii), (iii), etc., we have

$$P'_n=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+(2n-9)\,P_{n-5}+\ldots+3P_1+P_0'$$

$$=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+(2n-9)\,P_{n-5}+\ldots+3P_1$$

as $P'_0=0$ [See Ex. 1 after § 2·12]

when $n$ is odd.

$$P'_n=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+\ldots+5P_2+P'_1$$

$$=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+\ldots+P_0 \text{ as } P'_1=1=P_0$$

[See Ex. 1 after § 2·12]

Hence $P'_n=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+\ldots$
the last term of the series being $3P_1$ or $P_0$ according as $n$ is even or odd.

**§ 2·11. Christoffel's Summation Formula** : *To prove that*

$$\sum_{r=0}^{n}(2r+1)\,P_r(x)P_r(y)=(n+1)\,\frac{P_{n+1}(x)\,P_n(y)-P_{n+1}(y)\,P_n(x)}{(x-y)}$$

[Nagpur 85]

**Proof.** From Recurrence formula I, we have

$$(2r+1)\,xP_r(x)=(r+1)\,P_{r+1}(x)+rP_{r-1}(x) \qquad \ldots\text{(i)}$$

and

$$(2r+1)\,yP_r(y)=(r+1)\,P_{r+1}(y)+rP_{r-1}(y) \qquad \ldots\text{(ii)}$$

Multiplying (i) by $P_r(y)$ and (ii) by $P_r(x)$ and then subtracting we have

$$(2r+1)\,(x-y)\,P_r(x)\,P_r(y)=(r+1)\,[P_{r+1}(x)\,P_r(y)-P_{r+1}(y)\,P_r(x)]$$
$$-r\,[P_{r-1}(y)\,P_r(x)-P_{r-1}(x)\,P_r(y)].$$

Putting $r=0,1,2,3,\ldots(n-1),n$, we have

$$(x-y)\,P_0(x)\,P_0(y)=[P_1(x)\,P_0(y)-P_1(y)\,P_0(x)]+0 \qquad \ldots(A_0)$$

$$3\,(x-y)\,P_1(x)\,P_1(y)=2.[P_2(x)\,P_1(y)-P_2(y)\,P_1(x)]$$
$$-1.[P_0(y)\,P_1(x)-P_0(x)\,P_1(y)] \qquad \ldots(A_1)$$

$$5.(x-y)\,P_2(x)\,P_2(y)=3\,[P_3(x)\,P_2(y)-P_3(y)\,P_2(x)]$$
$$-2.[P_1(y)\,P_2(x)-P_1(x)\,P_2(y)] \qquad \ldots(A_2)$$

... ... ... ... ... ... ... ... ...

... ... ... ... ... ... ... ... ...

$$(2n-1)\,(x-y)\,P_{n-1}(x)\,P_{n-1}(y)$$
$$=n.[P_n(x).P_{n-1}(y)-P_n(y)\,P_{n-1}(x)]$$
$$-(n-1)\,[P_{n-2}(y)\,P_{n-1}(x)-P_{n-2}(x)\,P_{n-1}(y)] \qquad \ldots(A_{n-1})$$

$$(2n+1)\,(x-y)\,P_n(x)\,P_n(y)$$
$$=(n+1)\,[P_{n+1}(x)\,P_n(y)-P_{n+1}(y)\,P_n(x)]$$
$$-n.[P_{n-1}(y)\,P_n(x)-P_{n-1}(x)\,P_n(y)] \qquad \ldots(A_n)$$

Adding $(A_0)$, $(A_1)$, $(A_2)$ ..$(A_{n-1})$ and $(A_n)$, we have

$$(x-y)\sum_{r=0}^{n}(2r+1)\,P_r(x)\,P_r(y)$$
$$=(n+1)\,[P_{n+1}(x)\,P_n(y)-P_{n+1}(y)\,P_n(x)]$$

Hence $\displaystyle\sum_{r=0}^{n}(2r+1)\,P_r(x)\,P_r(y)$

$$=(n+1)\,\frac{P_{n+1}(x)\,P_n(y)-P_{n+1}(y)\,P_n(x)}{(x-y)}.$$

This is Christoffel's Summation Formula.

**§ 2·12. Rodrigue's Formula.** *To prove that*

$$P_n(x)=\frac{1}{n!\,2^n}\cdot\frac{d^n}{dx^n}(x^2-1)^n.$$

**[I.A.S. 79; Kanpur 83, 85, 86; GNUA 80; Agra 78, 84, 85; Meerut 78, 80 (S), 81 (P), 82 (P), 83, 85, 88, 89; Rohilkhand 83, 86, 88, 89; Jodhpur 84; B.H.U. 72]**

**Proof.** Let $y=(x^2-1)^n$.

Differentiating, $\dfrac{dy}{dx}=n(x^2-1)^{n-1}.2x$

$$\therefore\quad (x^2-1)\frac{dy}{dx}=2n\,xy.$$

Differentiating $(n+1)$ times by Leibnitz Theorem, we have

$$(x^2-1)\frac{d^{n+2}y}{dx^{n+2}}+(n+1).\frac{d^{n+1}y}{dx^{n+1}}2x-\frac{(n+1)\,n}{2!}\cdot\frac{d^ny}{dx^n}2$$

$$=2n\left[x.\frac{d^{n+1}y}{dx^{n+1}}+(n+1)\frac{d^ny}{dx^n}.1\right]$$

or $$(x^2-1)\frac{d^{n+2}y}{dx^{n+2}}+2x\frac{d^{n+1}y}{dx^{n+1}}-n(n+1)\frac{d^ny}{dx^n}=0$$

or $$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}}-2x\frac{d^{n+1}y}{dx^{n+1}}+n(n+1)\frac{d^ny}{dx^n}=0$$

Put $$Z=\frac{d^ny}{dx^n}$$

$$\therefore\quad (1-x^2)\frac{d^2Z}{dx^2}-2x\frac{dZ}{dx}+n(n+1)Z=0$$

which is Legendre's equation.

Hence its solution is

$Z=c\,P_n(x)$ where $c$ is a constant

or $$\frac{d^ny}{dx^n}=cP_n(x). \qquad ...(1)$$

Putting $x=1$, we have

$c=\left(\dfrac{d^ny}{dx^n}\right)_{x=1}$. Since $P_n(1)=1$ (see Ex. 3)

Now $y=(x^2-1)^n=(x+1)^n.(x-1)^n$.

Differentiating $n$ times by Leibnitz's theorem, we have

$$\frac{d^ny}{dx^n}=(x-1)^n.\frac{d^n}{dx^n}(x+1)^n+n.\left\{\frac{d^{n-1}}{dx^{n-1}}(x+1)^n\right\}.n(x-1)^{n-1}+\ldots$$

$$+n\left(\frac{d}{dx}(x+1)^n\right)\frac{d^{n-1}}{dx^{n-1}}(x-1)^n+(x+1)^n.\frac{d^n}{dx^n}(x-1)^n$$

$$=(x-1)^n\, n\,!+n.\frac{n\,!}{1\,!}(x+1)\, n\,(x-1)^{n-1}+\ldots$$

$$+n.n(x+1)^{n-1}\frac{n\,!}{1\,!}(x-1)+(x+1)^n.n\,!.$$

Putting $x=1.\left(\dfrac{d^ny}{dx^n}\right)_{x=1}=(1+1)^n.n\,!=2^n.n\,!=c$

$\therefore$ From (1), we have

$$P_n(x)=\frac{1}{c}\frac{d^ny}{dx^n}$$

or $$P_n(x)=\frac{1}{2^n.n\,!}\frac{d^n(x^2-1)^n}{dx^n}$$

This is Rodrigu's formula.

## EXAMPLES

**Ex. 1.** *Show that*

$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\dfrac{(3x^2-1)}{2},\ P_3(x)=\frac{1}{2}(5x^3-3x)$

$P_4(x)=(35x^4-30x^2+3)/8.$

**Proof.** We know that

$$\sum_{n=0}^{\infty} h^n\, P_n(x)=(1-2xh+h^2)^{-1/2}$$

$$=\{1-h\,(2x-h)\}^{-1/2}$$

$$=1+\frac{h}{2}(2x-h)+\frac{1.3}{2.4}h^2(2x-h)^2$$

$$+\frac{1.3.5}{2.4.6}h^3(2x-h)^3+\frac{1.3.5.7}{2.4.6.8}h^4(2x-h)^4+\ldots$$

or $$P_0(x)+h\,P_1(x)+h^2\,P_2(x)+h^3\,P_3(x)+h^4\,P_4(x)+\ldots$$
$$=1+x.h+\tfrac{1}{2}(3x^2-1)\,h^2+\tfrac{1}{2}(5x^3-3x)\,h^3$$
$$+\tfrac{1}{8}(35x^4-30x^2+3)+\ldots$$

Equating the coefficients of like powers of $h$, we have

$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{1}{2}(3x^2-1),\ P_3(x)=\frac{1}{2}(5x^3-3x)$

$P_4(x)=(35x^4-30x^2+3)/8$ etc.

**Ex. 2.** *Express* $P(x)=x^4+2x^3+2x^2-x-3$ *in terms of Legendre's polynomials.* **[Kanpur 88]**

**Sol.** From Ex. 1, we have

$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{1}{2}(3x^2-1),$

$P_3(x)=\frac{1}{2}(5x^3-3x),\ P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$

$\therefore$ From $P_4(x)=\dfrac{1}{8}(35x^4-30x^2+3),\ x^4=\dfrac{8}{35}P_4(x)+\dfrac{6}{7}x^2-\dfrac{3}{35}$

From $P_3(x)=\frac{1}{2}(5x^3-3x)$, $\quad x^3=\frac{2}{5}P_3(x)+\frac{3}{5}x$

From $P_2(x)=\frac{1}{2}(3x^2-1)$, $\quad x^2=\frac{2}{3}P_2(x)+\frac{1}{3}$

And $\quad x=P_1(x)$, $\quad 1=P_0(x)$.

Substituting the value of $x^4$, $x^3$, $x^2$, we have

$$\therefore \quad P(x)=\frac{8}{35}P_4(x)+\frac{6}{7}x^2-\frac{3}{35}+2x^3+2x^2-x-3$$

$$=\frac{8}{35}P_4(x)+2x^3+\frac{20}{7}x^2-x-\frac{108}{35}$$

$$=\frac{8}{35}P_4(x)+2\left[\frac{2}{5}P_3(x)+\frac{3}{5}x\right]+\frac{20}{7}x^2-x-\frac{108}{35}$$

$$=\frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{20}{7}x^2+\frac{1}{5}x-\frac{108}{35}$$

$$=\frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{20}{7}\left[\frac{2}{3}P_2(x)+\frac{1}{3}\right]+\frac{1}{5}x-\frac{108}{35}$$

$$=\frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{40}{21}P_2(x)+\frac{1}{5}x-\frac{224}{105}$$

$$=\frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{40}{21}P_2(x)+\frac{1}{5}P_1(x)-\frac{224}{105}P_0(x)$$

**Ans.**

**Ex. 3.** *Show that* (*i*) $P_n(1)=1$ **[Kanpur 85; Agra 74]**

(*ii*) $P_n(-x)=(-1)^n P_n(x)$.

**[Agra 89; Kanpur 85; GNDU Amritsar 87]**

*Hence deduce that* $\quad P_n(-1)=(-1)^n$.

**Proof.** (i) We know that

$$\sum_{n=0}^{\infty} h^n P_n(x)=(1-2xh+h^2)^{-1/2}$$

Putting $x=1$

$$\sum_{n=0}^{\infty} h^n P_n(1)=(1-2h+h^2)^{-1/2}$$

$$=(1-h)^{-1}$$

$$=1+h+h^2+\ldots+h^n+\ldots=\sum_{n=0}^{\infty} h^n$$

Equating the coefficient of $h^n$, we have $P_n(1)=1$,

(ii) We have, $(1-2xh+h^2)^{-1/2}=\sum_{n=0}^{\infty} h^n P_n(x)$

$\therefore \quad (1+2xh+h^2)^{-1/2}=\{1-2x(-h)+(-h)^2\}^{-1/2}$

$$=\sum_{n=0}^{\infty}(-h)^n P_n(x).$$

$$=\sum_{n=0}^{\infty}(-1)^n h^n P_n(x). \quad \ldots(1)$$

Again $\quad (1+2xh+h^2)^{-1/2}=\{1-2(-x)h+h\}^{2-1/2}$

$$=\sum_{n=0}^{\infty} h^n P_n(-x). \quad \ldots(2)$$

From (1) and (2), we have

$$\sum_{n=0}^{\infty} h^n P_n(-x\}=\sum_{n=0}^{\infty}(-1)^n h^n P_n(x).$$

Equating the coefficients of $h^n$,

$$\mathbf{P}_n(-\mathbf{x})=(-1)^n \mathbf{P}_n(\mathbf{x}).$$

**Deduction.** Putting $x=1$, we have

$$P_n(-1)=(-1)^n P_n(1)$$
$$=(-1)^n. \quad \text{Since } P_n(1)=1. \text{ See Ex. 2 (i)}$$

**[Kanpur 85]**

**Note.** If $n=2m$ then $P_n(-x)=(-1)^n P_n(x)$

gives $\quad P_{2m}(-x)=P_{2m}(x)$

and if $\quad n=2m+1$, $P_{2m+1}(-x)=-P_{2m+1}(x)$

*i.e.* if $\quad n$ is even $P_n(x)$ is even function of $x$.

and if $\quad n$ is odd $P_n(x)$ is odd function of $x$.

**Ex. 4.** *Prove that* $P_n(0)=0$, *for n odd*

*and* $P_n(0)=\dfrac{(-1)^{n/2}\, n\,!}{2^n\{(n/2)\,!\}^2}$, *for n even.* **[Meerut 84; Kanpur 83, 85]**

**Proof.** (i) We know that

$$P_n(x)=\frac{1.3.5\ldots(2n-1)}{n\,!}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}\right.$$
$$\left.+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}\ldots\right].$$

When $n=(2m+1)$, odd then

$$P_{2m+1}(x)=\frac{1.3.5\ldots\{2(2m+1)-1\}}{(2m+1)\,!}$$
$$\times\left[x^{2m+1}-\frac{(2m+1)(2m+1-1)}{2.\{2(2m+1)-1\}}x^{2m+1-2}+\ldots\right].$$

Putting $\quad x=0, \quad P_{2m+1}(0)=0.$

*i.e.* $\quad P_n(0)=0$, when $n$ is odd.

Also we have

$$\sum_{n=0}^{\infty} h^n P_n(x)=(1-2xh+h^2)^{-1/2}$$

$$\sum_{n=0}^{\infty} h^n P_n(0)=(1+h^2)^{-1/2}=\{1-(-h^2)\}^{-1/2}$$

$$=1+\tfrac{1}{2}.(-h^2)+\frac{1.3}{2.4}(-h^2)^2+\frac{1.3.5}{2.4.6}(-h^2)^3$$

$$+\ldots+\frac{1.3.5\ldots(2r-1)}{2.4\ldots2r}(-h^2)^r+\ldots$$

Here all powers of $h$ on the R.H.S. are even.

Equating the coefficient of $h^{2m}$ on both sides, we have

$$P_{2m}(0)=\frac{1.3.5\ldots(2m-1)}{2.4.6\ldots2m}(-1)^m=(-1)^m\frac{(2m)\,!}{2^{2m}(m\,!)^2}$$

*i.e.* when $n=2m$, then

$$P_n(0)=\frac{(-1)^{n/2}\,n\,!}{2^n\{(n/2)\,!\}^2}.$$ **Proved.**

**Ex. 5.** *Prove that*

$$C+\int P_n\,dx=\frac{P_{n+1}-P_{n-1}}{2n+1}.$$

**Proof.** From recurrence formula III, we have

$$(2n+1)\,P_n=P'_{n+1}-P'_{n-1}$$

or $$P_n=\frac{(P'_{n+1}-P'_{n-1})}{2n+1}.$$

Integrating, $$C+\int P_n\,dx=\frac{P_{n+1}-P_{n-1}}{2n+1}.$$ **Proved.**

**Ex. 6.** *Prove that*

$$P_0^2(x)+3P_1^2(x)+5P_2^2+\ldots+(2n+1).P_n^2(x)$$
$$=(n+1)\,[P_n(x)\,P'_{n+1}(x)-P_{n+1}(x)\,P'_n(x)]$$
$$=(n+1)^2\,P_n^2(x)+(1-x^2)\,\{P'_n(x)\}^2.$$

**Proof.** From Christoffel's summation formula, we have

$$(x-y)\sum_{r=0}^{n}(2r+1)\,P_r(x)\,P_r(y)$$
$$=(n+1)\,[P_{n+1}(x)\,P_n(y)-P_{n+1}(y)\,P_n(x)]$$

where $h$ is a small quantity

Putting $\quad y=x+h$

$$-h\sum_{r=0}^{n}(2r+1)\,P_r(x)\,P_r(x+h)$$
$$=(n+1)\,[P_{n+1}(x)\,P_n(x+h)-P_{n+1}(x+h)\,P_n(x)].$$

Expanding by Taylor's theorem, we have

$$-h\sum_{r=0}^{n}(2r+1)\,P_r(x)\,\{P_r(x)+hP'_r(x)+\ldots\}$$

$$=(n+1)\left[P_{n+1}(x)\left\{P_n(x)+hP'_n(x)+\frac{h^2}{2!}P''_n(x)+\ldots\right\}\right.$$

$$\left.-\left\{P_{n+1}(x)+hP'_{n+1}(x)+\frac{h^2}{2!}P''_{n+1}(x)+\ldots\right\}P_n(x)\right]$$

$$=-h(n+1)\,[\{P_n(x)\,P'_{n+1}(x)-P'_n(x)\,P_{n+1}(x)+h(\ldots)+\ldots]$$

or $$\sum_{r=0}^{n}(2r+1)\,P_r(x)\,\{P_r(x)+hP'_r(x)+\ldots\}$$

$$=(n+1)\,[P_n(x).P'_{n+1}(x)-P'_n(x)\,P_{n+1}(x)+h\,(\ldots)+\ldots].$$

Taking limit as $h\to 0$, we have

$$\sum_{r=0}^{n}(2r+1)\,P_r^2(x)=(n+1)\,[P_n(x)\,P'_{n+1}(x)-P'_n(x)\,P_{n+1}(x)]$$

or $$P_0^2(x)+3P_1^2(x)+5P_2^2(x)+\ldots+(2n+1)\,P_n^2(x)$$

$$=(n+1)\,[P_n(x)\,P'_{n+1}(x)-P_{n+1}(x)\,P'_n(x)].$$ **Proved.**

Again $(n+1)^2\,P_n^2(x)+(1-x^2)\,P'^2_n(x)$

$$=(n+1)\,P_n(x)\,[(n+1)\,P_n(x)]+P'_n(x)\,[(1-x^2)\,P'_n(x)]$$

$$=(n+1)\,P_n(x)\,[P'_{n+1}(x)-xP'_n(x)]$$

$$+P'_n(x)\,[(n+1)\,\{xP_n(x)-P_{n+1}(x)\}]$$

from recurrence formula IV and VI

$$=(n+1)\,[P_n(x)\,P'_{n+1}(x)-P_{n+1}(x)\,P'_n(x)].$$ **Hence Proved.**

**Ex. 7.** *Prove that* $(1-2xZ+Z^2)^{-1/2}$ *is a solution of the equation*

$$Z\,\frac{\partial^2\,(Zv)}{\partial Z^2}+\frac{\partial}{\partial x}\left\{(1-x^2)\,\frac{\partial v}{\partial x}\right\}=0.$$

**[Ravishankar 85; Agra 80; Kanpur 85]**

**Proof.** We have

$$v=(1-2xZ+Z^2)^{-1/2}=\sum_{n=0}^{\infty}Z^n\,P_n$$

or $$Zv=\sum_{n=0}^{\infty}Z^{n+1}\,P_n$$

$$\therefore\ Z\,\frac{\partial^2}{\partial Z^2}\,(Zv)=\sum_{n=0}^{\infty}(n+1)\,nZ^n\,P_n$$

Also $$\frac{\partial v}{\partial x}=\sum_{n=0}^{\infty} Z^n P'_n.$$

$$\therefore \quad \frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial v}{\partial x}\right\}=\frac{\partial}{\partial x}\left((1-x^2)\sum_{n=0}^{\infty} Z^n P'_n\right)$$

$$=(1-x^2)\sum_{n=0}^{\infty} Z^n P''_n-2x\sum_{n=0}^{\infty} Z^n P'_n$$

Substituting in the L.H.S. of the given equation, we have

$$Z.\frac{\partial^2 (Zv)}{\partial Z^2}+\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial v}{\partial x}\right\}$$

$$=\sum_{n=0}^{\infty}[(n+1)\,nZ^n P_n+(1-x^2)\,Z^n P''_n-2xZ^n P'_n]$$

$$=\sum_{n=0}^{\infty} Z^n\,[(1-x^2)\,P''_n-2xP'_n+n\,(n+1)\,P_n]$$

$=0.$ Since $P_n$ is a solution of Legendre's equation.

**Ex. 8.** *Prove that*

$$\frac{1+Z}{Z\sqrt{\{(1-2xZ+Z^2)\}}}-\frac{1}{Z}=\sum_{n=0}^{\infty}[P_n(x)+P_{n+1}(x)]\,Z^n.$$

**Proof.** $$\frac{1+Z}{Z\sqrt{\{(1-2xZ+Z^2)\}}}-\frac{1}{Z}$$

$$=\frac{1}{Z}\,(1-2xZ+Z^2)^{-1/2}+(1-2xZ+Z^2)^{-1/2}-\frac{1}{Z}$$

$$=\frac{1}{Z}\sum_{n=0}^{\infty} Z^n P_n(x)+\sum_{n=0}^{\infty} Z^n P_n(x)-\frac{1}{Z}$$

$$=\frac{1}{Z}\left[P_1(x)+\sum_{n=1}^{\infty} Z^n P_n(x)\right]+\sum_{n=0}^{\infty} Z^n P_n(x)-\frac{1}{Z}$$

$$=\frac{1}{Z}+\frac{1}{Z}\sum_{n=1}^{\infty} Z^n P_n(x)+\sum_{n=0}^{\infty} Z^n P_n(x)-\frac{1}{Z} \qquad \text{since } P_0(x)=1$$

$$=\sum_{n=1}^{\infty} Z^{n-1}\,P_n(x)+\sum_{n=0}^{\infty} Z^n P_n(x)$$

$$=\sum_{n=0}^{\infty} Z^n P_{n+1}(x)+\sum_{n=0}^{\infty} Z^n P_n(x)$$

$$=\sum_{n=0}^{\infty} [P_{n+1}(x)+P_n(x)]\, Z^n.$$ **Proved.**

**Ex. 9.** *Show that*

$$\frac{1-Z^2}{(1-2xZ+Z^2)^{3/2}}=\sum_{n=0}^{\infty} (2n+1)\, P_n(x) Z^n.$$

**[Rohilkhand 80; Jiwaji 82]**

**Proof.** We have

$$(1-2x\,Z+Z^2)^{-1/2}=\sum_{n=0}^{\infty} Z^n\, P_n(x) \quad \text{...(i)}$$

Differentiating w.r.t. $Z$ we have

$$(x-Z)\,(1-2x\,Z+Z^2)^{-3/2}=\sum_{n=0}^{\infty} nZ^{n-1}\, P_n(x).$$

$$\therefore\quad 2(x-Z)\,Z\,(1-2x\,Z+Z^2)^{-3/2}=\sum_{n=0}^{\infty} 2n\, Z^n\, P_n(x). \quad \text{...(ii)}$$

Adding (i) and (ii), we have

$$\frac{1-2x\,Z+Z^2+2\,(x-Z)\,Z}{(1-2x\,Z+Z^2)^{3/2}}=\sum_{n=0}^{\infty} (2n+1)\, Z^n\, P_n(x)$$

$$\frac{1-Z^2}{(1-2x\,Z+Z^2)^{3/2}}=\sum_{n=0}^{\infty} (2n+1)\, Z^n\, P_n(x).$$ **Proved.**

**Ex. 10.** *Prove that*

$$P'_{n+1}+P'_n=P_0+3P_1+5P_2+\ldots+(2n+1)\, P_n=\sum_{r=1}^{n} (2r+1)\, P_r(x).$$

**[Rohilkhand 82; Kanpur 83; Meerut 81, 82, 86; Agra 82, 87]**

**Proof.** Writing Recurrence formula III, we have

$$(2n+1)\, P_n=P'_{n+1}-P'_{n-1}.$$

Putting $n=1, 2, 3, \ldots$ we get

$$3P_1=P'_2-P'_0$$
$$5P_2=P'_3-P'_1$$
$$7P_3=P'_4-P'_2$$
$$\ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots$$
$$(2n-3)\, P_{n-2}=P'_{n-1}-P'_{n-3}$$

$$(2n-1)\ P_{n-1}=P'_n-P'_{n-2}$$
$$(2n+1)\ P_n=P'_{n+1}-P'_{n-1}.$$

Adding all, we get

$$3P_1+5P_2+\ldots+(2n+1)\ P_n=P'_n+P'_{n+1}-P'_0-P'_1$$
$$=P'_n+P'_{n+1}-0-P_0.$$

Since $P_0=1$ and $P_1=x$. $\quad\therefore\quad P'_1=1=P_0$

Hence $P_0+3P_1+5P_2+\ldots+(2n+1)\ P_n=P'_{n+1}+P'_n$.

**Ex. 11.** *Prove that*

$$\int_{-1}^{+1}(x^2-1)\ P_{n+1}\ P'_n\ dx=\frac{2n\ (n+1)}{(2n+1)\ (2n+3)}.$$

**[Jodhpur 84; Kanpur 80; Agra 87; Raj. 81]**

**Proof.** From Recurrence formula V, we have

$$(x^2-1)\ P'_n=n\ (xP_n-P_{n-1}).$$

$$\therefore\quad \int_{-1}^{+1}(x^2-1)\ P_{n+1}\ P'_n\ dx$$
$$=\int_{-1}^{+1} n\ (x\ P_n-P_{n-1})\ P_{n+1}\ dx$$

the other integral being zero since

$$=n\int_{-1}^{+1} xP_nP_{n+1}\ dx,\qquad \int_{-1}^{+1}P_mP_n\ dx=0,\ \text{if}\ m\neq n$$

$$=n\int_{-1}^{+1}\frac{(n+1)\ P_{n+1}+n\ P_{n-1}}{2n+1}\ P_{n+1}\ dx \quad \text{from Rec. formula I}$$

$$=\frac{n\ (n+1)}{2n+1}\int_{-1}^{+1}P^2_{n+1}\ dx+\frac{n^2}{2n+1}\int_{-1}^{+1}P_{n-1}\ P_{n+1}\ dx$$

$$=\frac{n\ (n+1)}{(2n+1)}\cdot\frac{2}{2\ (n+1)+1}+0 \qquad \text{from § 2·7 (i), (ii)}$$

$$=\frac{2n\ (n+1)}{(2n+1)\ (2n+3)}.$$

**Ex. 12.** *Prove that*

(*i*) $\displaystyle\int_{-1}^{+1}P_n(x)\ dx=0,\ n\neq 0$ **[Meerut 73]**

*and* (*ii*) $\displaystyle\int_{-1}^{+1}P_0(x)\ dx=2.$ **[Rohilkhand 89]**

**Proof.** From Rodrigue's formula, we have

$$P_n(x)=\frac{1}{2^n\ n\ !}\cdot\frac{d^n}{dx^n}\ (x^2-1)^n$$

$$\therefore\quad \int_{-1}^{+1}P_n(x)\ dx=\frac{1}{2^n\ n\ !}\int_{-1}^{+1}\frac{d^n}{dx^n}\ (x^2-1)^n\ dx$$

$$=\frac{1}{2^n\, n!}\left\{\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right\}_{-1}^{+1}$$

Now $\dfrac{d^{n-1}}{dx^{n-1}}(x^2-1)^n=\dfrac{d^{n-1}}{dx^{n-1}}(x+1)^n(x-1)^n$ ...(1)

$$=(x+1)^n\frac{d^{n-1}}{dx^{n-1}}(x-1)^n$$

$$+(n-1).n(x+1)^{n-1}\frac{d^{n-2}}{dx^{n-2}}(x-1)^n+\ldots$$

$$+(x-1)^n\frac{d^{n-1}}{dx^{n-1}}(x+1)^n$$

$$=(x+1)^n\frac{n!}{1!}(x-1)$$

$$+n(n-1)(x+1)^{n-1}\frac{n!}{2!}(x-1)^2+\ldots$$

$$\ldots+(x-1)^n\, n!\,(x+1)$$

$=0$ when $x=-1$ or $1$

since each term contains $(x-1)$ and $(x+1)$

$\therefore$ from (1), $\displaystyle\int_{-1}^{+1}P_n(x)\,dx=0.$

(ii) We know that $P_0(x)=1.$

$$\therefore\quad \int_{-1}^{+1}P_0(x)\,dx=\int_{-1}^{+1}dx=\left\{x\right\}_{-1}^{+1}=2.$$

**Ex. 13.** *Prove that if m is an integer less than n.*

$$\int_{-1}^{+1}x^m P_n(x)\,dx=0$$

**[Rohilkhand 86; Kanpur 87]**

*and* $$\int_{-1}^{+1}x^n P_n(x)\,dx=\frac{2^{n+1}(n!)^2}{(2n+1)!}.$$

**Proof.** Let

$$I=\int_{-1}^{+1}x^m P_n(x)\,dx$$

$$=\int_{-1}^{+1}x^m\frac{1}{2^n\,n!}\frac{d^n}{dx^n}(x^2-1)^n\,dx \text{ By Rodrigue's formula}$$

$$=\frac{1}{2^n\,n!}\int_{-1}^{+1}x^m.\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$=\frac{1}{2^n\,n!}\left[\left\{x^m.\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx\,dx\right\}_{-1}^{+1}\right.$$

$$\left.-\int_{-1}^{+1}mx^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx\right]$$

$$=\frac{(-1)\,m}{2^n.n\,!}\int_{-1}^{+1} x^{m-1}\,\frac{d^{n-1}}{dx^{n-1}}\,(x^2-1)^n\,dx$$

$$\left[\text{since } \left\{\frac{d^{n-1}}{dx^{n-1}}.(x^2-1)^n\right\}_{-1}^{+1}=0, \text{ see Ex. 12}\right]$$

Proceeding similarly, we have

$$I=\frac{(-1)^m.m\,!}{2^n.n\,!}\int_{-1}^{+1} x^0\,\frac{d^{n-m}}{dx^{n-m}}\,(x^2-1)^n.dx$$

$$=\frac{(-1)^m\,m\,!}{2^n.n\,!}\left[\frac{d^{n-m-1}}{dx^{n-m-1}}\,(x^2-1)^n\right]_{-1}^{+1}$$

$$=0 \quad [\text{can be shown easily as } n>m+1]$$

Again if $m=n$, then

$$\int_{-1}^{+1} x^n\,P_n(x)\,dx=\frac{(-1)^n.n\,!}{2^n\,n\,!}\int_{-1}^{+1}\frac{d^{n-n}}{dx^{n-n}}\,(x^2-1)^n\,dx$$

$$=\frac{(-1)^n}{2^n}.\int_{-1}^{+1}(x^2-1)^n\,dx$$

$$=\frac{2}{2^n}\int_0^1(1-x^2)^n\,dx$$

$$=\frac{2}{2^n}\int_0^{\pi/2}\cos^{2n+1}\theta\,d\theta. \quad \text{Put } x=\sin\theta$$

$$=\frac{2}{2^n}\cdot\frac{\Gamma(n+1)\,\Gamma(\tfrac{1}{2})}{2\Gamma\left\{\dfrac{(2n+3)}{2}\right\}}$$

$$=\frac{2}{2^n}\cdot\frac{n\,!\,\sqrt{\pi}}{2\left\{\dfrac{2n+1}{2}\right\}\left\{\dfrac{2n-1}{2}\right\}\left\{\dfrac{2n-3}{2}\right\}\cdots\dfrac{5}{2}\cdot\dfrac{3}{2}\cdot\dfrac{1}{2}\sqrt{\pi}}$$

$$=\frac{2\,(n\,!)^2}{2^n.2.\left\{\dfrac{2n+1}{2}\right\}\left\{\dfrac{2n}{2}\right\}\left\{\dfrac{2n-1}{2}\right\}\left\{\dfrac{2n-2}{2}\right\}\cdots\dfrac{3}{2}\cdot\dfrac{2}{2}\cdot\dfrac{1}{2}}$$

$$=\frac{(n\,!)^2}{2^n.\dfrac{(2n+1)\,!}{2^{2n+1}}}=\frac{2^{n+1}\,(n\,!)^2}{(2n+1)\,!}$$

**Proved.**

**Ex. 14.** *Prove that*

$$\int_{-1}^{+1} x^2\,P_{n+1}\,P_{n-1}\,dx=\frac{2n\,(n+1)}{(2n-1)\,(2n+1)\,(2n+3)}$$

**[Jodhpur 81; Kanpur 85; Meerut 80 (S), 90; Raj. 79, 83; Agra 82]**

**Proof.** From Recurrence formula I, we have

$$(2n+1)\,x\,P_n=(n+1)\,P_{n+1}+n\,P_{n-1}.$$

Replacing $n$ by $(n-1)$ and $(n+1)$ respectively, we have

$$(2n-1)\,x\,P_{n-1}=nP_n+(n-1)\,P_{n-2}$$

and $$(2n+3)\,x\,P_{n+1}=(n+2)\,P_{n+2}+(n+1)\,P_n$$

Multiplying

$$(2n-1)(2n+3)\,x^2 P_{n+1} P_{n-1} = n(n+1) P_n^2 + n(n+2) P_n P_{n+2} + (n-1)(n+2) P_{n-2} P_{n+2} + (n-1)(n+1) P_{n-2} P_n$$

Integrating between the limits $-1$ to $+1$, we have

$$(2n-1)(2n+3)\int_{-1}^{+1} x^2 P_{n+1} P_{n-1}\,dx = n(n+1)\int_{-1}^{+1} P_n^2\,dx$$

all other integrals being zero

$$= n(n+1)\frac{2}{(2n+1)}$$

$$\therefore \quad \int_{-1}^{+1} x^2 P_{n+1} P_{n-1}\,dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

**Ex. 15.** *Prove that*

$$\int_{-1}^{+1} (1-x^2)\, P_m' P_n'\,dx = 0$$

*where m and n are distinct positive integers.*

**Proof.** $\int_{-1}^{+1} (1-x^2)\, P_m' P_n'\,dx$

$$= \Big[(1-x^2)\, P_m' P_n\Big]_{-1}^{+1} - \int_{-1}^{+1}\left[P_n \frac{a}{dx}\{(1-x^2)\, P'_m\}\right] dx.$$

(Integrating by parts taking $P_n$ as IInd function)

$$= -\int_{-1}^{+1}\left[P_n \frac{d}{dx}\{(1-x^2)\, P'_m\}\right] dx. \qquad \ldots(1)$$

Now since $P_m$ is the solution of the Legendre's equation

$$\therefore \quad (1-x^2)\, P''_m - 2xP'_m + m(m+1) P_m = 0$$

or

$$\frac{d}{dx}[(1-x^2)P'_m] = -m(m+1) P_m.$$

Hence from (1), we have

$$\int_{-1}^{+1} (1-x^2)\, P'_m P'_n\,dx = -\int_{-1}^{+1} [-P_n \,.\, m(m+1) P_m]\,dx$$

$$= m(m+1)\int_{-1}^{+1} P_n P_m\,dx$$

$$= 0. \quad \text{Since } m \neq n.$$

**Ex. 16.** *Prove that*

$$\int_{-1}^{+1} (1-x^2)(P'_n)^2\,dx = \frac{2n(n+1)}{2n+1}.$$

**[Rohilkhand 82]**

**Proof.** From Christoffel's result, we have

$$P'_n = (2n-1) P_{n-1} + (2n-5) P_{n-3} + (2n-9) P_{n-5} + \ldots \qquad \ldots(\text{i})$$

Also from Beltrami's result, we have

$$(1-x^2)\, P'_n = \frac{n\,(n+1)}{2n+1}\,(P_{n-1}-P_{n+1}). \qquad \text{...(ii)}$$

Multiplying (i) and (ii) and then integrating between the limits $-1$ to $+1$, we have

$$\int_{-1}^{+1} (1-x^2)\,(P'_n)^2\, dx$$

$$=\frac{n\,(n+1)}{2n+1}\int_{-1}^{+1} [(2n-1)\, P^2_{n-1}-(2n-1)\, P_{n-1}\, P_{n+1}$$
$$+(2n-5)\, P_{n-1}\, P_{n+1}...]\, dx$$

$$=\frac{n\,(n+1)\,(2n-1)}{2n+1}\int_{-1}^{+1} P^2_{n-1}\, dx$$

$\left[\text{since all other integrals are zero by the property } \int_{-1}^{+1} P_m P_n\, dx=0,\ m\neq n\right]$

$$=\frac{n\,(n+1)\,(2n-1)}{2n+1}\cdot\frac{2}{2\,(n-1)+1}$$

$$=\frac{2n\,(n+1)}{2n+1}.$$

**Proved.**

**Ex. 17.** *Prove that*

$$\int_{-1}^{+1} (P'_n)^2\, dx = n\,(n+1).$$

**[Agra 89; Meerut 79; Kanpur 83, 84]**

**Proof.** From Christoffel's expansion, we have

$$P'_n=(2n-1)\, P_{n-1}+(2n-5)\, P_{n-3}+(2n-9)\, P_{n-5}+... \qquad \text{...(1)}$$

the last term is $P_0$ *i.e.* 1 or $3\,P_1$ *i.e.* $3x$ according as $n$ is odd or even.

*When n is odd,* let $m$ be the number of terms on the R.H.S. of (1)

then last Coeff. $1=(2n-1)+(m-1).(-4)$ ; $\therefore\ m=\dfrac{n+1}{2}$

and *when n is even,* let $m'$ be the number of terms on the R.H.S. of (1)

then last Coeff. $3=(2n-1)+(m'-1)\,(-4)$. $\therefore\ m'=n/2.$

Now

$$(P'_n)^2=(2n-1)^2\, P^2_{n-1}+(2n-5)^2\, P^2_{(n-3)}+...$$
$$+2\,(2n-1)\,(2n-5)\, P_{n-1}\, P_{n-3}+...$$

$$\therefore \int_{-1}^{+1} (P'_n)^2\, dx=(2n-1)^2\int_{-1}^{+1} (P_{n-1})^2\, dx+(2n-5)^2\int_{-1}^{+1} P^2_{n-3}dx+...$$

Other integrals are zero by § 2·7 (i)

$$=(2n-1)^2 \frac{2}{2(n-1)+1}+(2n-5)^2 \cdot \frac{2}{2(n-3)+1}+(2n-9)^2 \frac{2}{2(n-5)+1}+\dots$$

$$=2\,[(2n-1)+(2n-5)+(2n-9)+\dots] \qquad \dots(2)$$

the last term being 1 or 3 according as $n$ is odd or even.

**Case I.** When $n$ is even, no of terms on the R.H.S. of (2) is $n/2$.

$$\therefore \int_{-1}^{+1} (P'_n)^2\, dx=2\cdot\frac{1}{2}\cdot\frac{n}{2}\left[2(2n-1)+\left(\frac{n}{2}-1\right)(-4)\right]=n\,(n+1).$$

**Case II.** When $n$ is odd, no. of terms on the R.H.S. of (2) is $\frac{n+1}{2}$.

$$\therefore \int_{-1}^{+1} (P'_n)^2\, dx=2\cdot\frac{1}{2}\left(\frac{n+1}{2}\right)\left[2\cdot(2n-1)+\left\{\frac{n+1}{2}-1\right\}(-4)\right]$$
$$=n\,(n+1).$$

Hence

$$\int_{-1}^{+1} (P'_n)^2\, dx=n\,(n+1).$$ **Proved.**

**Ex. 18.** *Show that all the roots of $P_n(x)=0$ are real and lie between $-1$ and $+1$.* [Agra 81; Raj. 78; Kanpur 84, 87]

**Proof.** We have $P_n(x)=\frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n$.

Now $(x^2-1)^n=(x-1)^n.(x+1)^n$.

Hence $(x^2-1)^n$ vanishes $n$ times at $x=+1$ and $n$ times at $x=-1$. Therefore by the theory of equations, $\frac{d^n}{dx^n}(x^2-1)^n$ will have $n$ roots all real and lying between $-1$ and $+1$.

Hence $P_n(x)=\frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2-1)^n=0$ has $n$ roots all real and lying between $-1$ and $+1$. **Proved.**

**Ex. 19.** *Prove that all the root of $P_n(x)=0$ are distinct.*

**Proof.** If the roots of $P_n(x)=0$ are not all different, then at least two of them must be equal.

Let $\alpha$ be their common value

$$\therefore \qquad P_n(\alpha)=0 \qquad \dots(1)$$

and

$$P'_n(\alpha)=0. \qquad \dots(2)$$

But $P_n(x)$ is the solution of Legendre's equation.

$$\therefore \quad (1-x^2)\frac{d^2}{dx^2}P_n(x)-2x\frac{d}{dx}P_n(x)+n(n+1)P_n(x)=0.$$

Differentiating $r$ times by Leibnitz's theorem, we have

$$\left[(1-x^2)\frac{d^{r+2}}{dx^{r+2}}P_n(x)-2x\,{}^rC_1\frac{d^{r+1}}{dx^{r+1}}P_n(x)-2\,{}^rC_2\frac{d^r}{dx^r}P_n(x)\right]$$

$$-2\left[x\frac{d^{r+1}}{dx^{r+1}}P_n(x)+1.{}^rC_1\frac{d^r}{dx^r}P_n(x)\right]+n(n+1)\frac{d^r}{dx^r}P_n(x)=0$$

or $$(1-x^2)\frac{d^{r+2}}{dx^{r+2}}P_n(x)-2x({}^rC_1+1)\frac{d^{r+1}}{dx^{r+1}}P_n(x)$$

$$-\{2.{}^rC_2+2.{}^rC_1-n(n+1)\}\frac{d^r}{dx^r}P_n(x)=0 \quad \ldots(3)$$

Putting $r=0$ and $x=\alpha$, we have

$$(1-\alpha^2)\left[\frac{d^2}{dx^2}P_n(x)\right]_{x=\alpha}-2\alpha\left[\frac{d}{dx}P_n(x)\right]_{x=\alpha}+n(n+1)P_n(\alpha)=0$$

or $$P_n''(\alpha)=0 \text{ by (1) and (2).}$$

Similarly writing $r=1, 2, 3, \ldots$ in (3) and simplifying stepwise we have

$$P_n'''(\alpha)=0=P_n^{iv}(\alpha)=\ldots=P^n{}_n(\alpha).$$

But since $P_n(x)=\dfrac{1.3\ldots(2n-1)}{n\,!}\left[x^n-\dfrac{n(n-1)}{2.(2n-1)}x^{n-2}\ldots\right]$

$$\therefore \quad P^n{}_n(\alpha)=\frac{1.3\ldots(2n-1)}{n\,!}\,n\,!$$

Hence $P^n{}_n(\alpha)\neq 0.$

Therefore our assumption that $P_n(x)=0$ has a repeated root is not correct.

Hence all the roots of $P_n(x)=0$ are distinct. **Proved.**

**Ex. 20.** *Prove that* (*i*) $P_n'(1)=\frac{1}{2}n(n+1)$

(*ii*) $P_n'(-1)=(-1)^{n-1}\frac{1}{2}n(n+1)$.

**Proof.** $P_n(x)$ satisfies Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)P_n(x)=0$$

$$\therefore \quad (1-x^2)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0 \quad \ldots(1)$$

(i) putting $x=1$, in (1), we have

$$-2P_n'(1)+n(n+1)P_n(1)=0$$

$$\therefore \quad P'_n(1)=\tfrac{1}{2}n(n+1), \text{ since } P_n(1)=1$$ **Proved.**

Putting $x=-1$ in (1), we have

$$2P'_n(-1)+n(n+1)P_n(-1)=0$$

or $$P'_n(-1)=-\tfrac{1}{2}n(n+1)P_n(-1)$$

$$=(-1)^{n-1}.\tfrac{1}{2}n(n+1) \text{ since } P_n(-1)=(-1)^n$$

**Proved.**

**Ex. 21.** *Prove that*

$$\int_0^1 P_n(x)\,dx=\frac{(-1)^{(n-1)/2}\,(n-1)\,!}{2^n\,\{(n+1)/2\}\,!\,\{(n-1)/2\}\,!}$$ [Meerut 76]

*when n is odd.*

**Proof.** From Recurrence formula, we have

$$P_n(x)=\frac{1}{(2n+1)}\,[P'_{n+1}\,(x)-P'_{n-1}\,(x)]$$

$$\therefore\quad \int_0^1 P_n\,(x)\,dx=\frac{1}{(2n+1)}\int_0^1 \{P'_{n+1}\,(x)-P'_{n-1}\,(x)\}\,dx$$

$$=\frac{1}{(2n+1)}\Big[P_{n+1}\,(x)-P_{n-1}\,(x)\Big]_0^1$$

$$=\frac{1}{(2n+1)}\,[P_{n+1}\,(1)-P_{n-1}\,(1)-P_{n+1}\,(0)+P_{n-1}\,(0)]$$

$$=\frac{1}{(2n+1)}\Big[1-1-(-1)^{(n+1)/2}\,\frac{(n+1)\,!}{2^{n+1}[\{(n+1)/2\}!]^2}+(-1)^{(n-1)/2}\,\frac{(n-1)\,!}{2^{n-1}[\{(n-1)/2\}\,!]^2}\Big]$$

(By Ex. 4 (ii) Part $\because$ $(n+1)$ and $(n-1)$ are even when $n$ is odd)

$$=\frac{1}{(2n+1)}.(-1)^{(n-1)/2}.\frac{(n-1)\,!}{[\{(n-1)/2\}!]^2}.\frac{1}{2^{n-1}}\left[(-1)^2\,\frac{(n+1)\,n}{2^2\{(n+1)/2\}^2}+1\right]$$

$$=\frac{1}{(2n+1)}\,(-1)^{(n-1)/2}\,\frac{(n-1)\,!}{2^{n-1}[\{(n-1)/2\}!]^2}\left[\frac{n}{n+1}+1\right]$$

$$=(-1)^{(n-1)/2}\,\frac{(n-1)\,!}{2^{n-1}\,\{(n-1)/2\}\,!\,\{n-1)/2\}\,!}.\frac{1}{(n+1)}$$

$$=(-1)^{(n-1)/2}\,\frac{(n-1)\,!}{2^{n-1}\{(n-1)/2\}!\,.2(n+1)/2.\{(n-1)/2\}!}$$

$$=\frac{(-1)^{(n-1)/2}\,(n-1)\,!}{2^n\,\{(n+1)/2\}\,!\,\{(n-1)/2\}\,!}$$ **Proved**

**Ex. 22.** *If* $x>1$, *show that* $P_n\,(x)<P_{n+1}\,(x)$.

**Proof.** We shall prove this by the method of induction *i.e.*, we shall assume that $P_{n-1}(x)<P_n(x)$ and use this to prove that $P_n\,(x)<P_{n+1}\,(x)$. Then since the result is true for $n=0$ as $P_0\,(x)=1$, $P_1\,(x)=x$, so that $P_0\,(x)<P_1\,(x)$.

$\therefore$ It will be true for all values of $n$.

By Rodrigue's formula I, we have

$$P_n\,(x)=\frac{1}{n\,!\,2^n}.\frac{d^n}{dx^n}\,(x^2-1)^n.$$

If $x > 1$, $P_n(x) > 0$ for all values of $n$.

From recurrence formula I, we have

$$(2n+1)\, xP_n(x) = (n+1)\, P_{n+1} + n\, P_{n-1}$$

or $$(2n+1)\, x = (n+1)\, \frac{P_{n+1}}{P_n} + n\, \frac{P_{n-1}}{P_n}$$

or $$\frac{P_{n+1}}{P_n} = \frac{(2n+1)}{(n+1)}\, x - \frac{n}{n+1}\, \frac{P_{n-1}}{P_n}$$

$$< \frac{(2n+1)}{(n+1)} - \frac{n}{n+1}$$ since $x > 1$ and we already assumed that $\frac{P_{n-1}}{P_n} < 1$

or $$\frac{P_{n+1}}{P_n} > \frac{n+1}{n+1}$$

or $$\frac{P_{n+1}}{P_n} > 1$$

or $$P_{n+1} > P_n.$$

Hence assuming $P_{n-1} < P_n$, *i.e.* $P_n > P_{n-1}$

we have proved $P_{n+1} > P_n$,

and since this is also true for $n=0$.

$\therefore$ It is true for $n=1$ and so on.

Hence this is true for all values of $n$.

## § 2·13. Even and odd Functions.

**Definition.**

**Even function :** $f(x)$ is said to be an even function of $x$ if $f(-x) = f(x)$.

If an even function $f(x)$ is expressed as a series of functions so that

$$f(x) = f_1(x) + f_2(x) + \ldots + f_n(x) + \ldots$$

Then all $f_1(x)$, $f_2(x)$ ..etc., must be even function of $x$. For if any of these is an odd function of $x$, then $f(-x) \neq f(x)$.

**Odd function :** $f(x)$ is said to be an odd function of $x$ if

$$f(-x) = -f(x).$$

If an odd function $f(x)$ is expressed as a series of functions so that

$$f(x) = f_1(x) + f_2(x) + \ldots + f_n(x) + \ldots$$

none of $f_n$'s may be an even function of $x$.

## § 2·14. Expansion of $x^n$ in Legendre's Polynomials.

**[Kanpur 84]**

Let $$x^n = a_n P_n(x) + a_{n-2}\, P_{n-2}(x) + a_{n-4}\, P_{n-4}(x) + \ldots + a_r\, P_r(x) + \ldots \qquad \ldots(1)$$

where $P_n(x)$, $P_{n-2}(x)$ are all even or odd functions of $x$ according as $x^n$ is an even or odd function.

Since $P_n(x)$ contains terms of degree $n$ and lower, we cannot have in the expansion of $x^n$ any $P$ with suffix higher than $n$

Also $P_{n-1}(x)$, $P_{n-3}(x)$......cannot occur in this expansion as they are odd or even functions of $x$ if $x^n$ is even and odd function of $x$ respectively.

Now we have to determine the values of the coefficients $a_n, a_{n-1}$......etc. For this let us multiply both sides of (1) by $p_r(x)$ and then integrate between the limits $-1$ to $+1$.

Thus we get $\int_{-1}^{+1} x^n . P_r(x) = a_r \int_{-1}^{+1} P_r^2(x)\, dx$

[Since all other integrals on the L.H.S. are zero by the property

$$\int_{-1}^{1} P_m P_n\, dx = 0, \text{ if } m \neq n\Big]$$

$$= a_r . \frac{2}{2r+1}$$

or
$$a_r = \frac{2r+1}{2} \int_{-1}^{+1} x^n P_r(x)\, dx \qquad \text{...(2)}$$

$$= \frac{(2r+1)}{2} \int_{-1}^{+1} x^n . \frac{1}{2^r . r!} \frac{d^r}{dx^r} (x^2-1)^r\, dx$$

By Rodrigue's formula § 2·12

$$= \frac{(2r+1)}{2^{r+1}\, r!} \int_{-1}^{+1} x^n . \frac{d^r}{dx^r} (x^2-1)^r\, dx.$$

Integrating by parts taking $x^n$ as first function

$$= \frac{(2r+1)}{2^{r+1}.r!} \left[ \left\{ x^n \frac{d^{r-1}}{dx^{r-1}} (x^2-1)^r \right\}_{-1}^{+1} - n \int x^{n-1} \frac{d^{r-1}}{dx^{r-1}} (x^2-1)^r\, dx \right]$$

$$= (-1) \frac{(2r+1)}{2^{r+1}\, r!} n \int_{-1}^{+1} x^{n-1} \frac{d^{r-1}}{dx^{r-1}} (x^2-1)^r\, dx$$

$$= (-1)^2 \frac{(2r+1)}{2^{r+1}\, r!} n(n-1) \int_{-1}^{+1} x^{n-2} \frac{d^{r-2}}{dx^{r-2}} (x^2-1)^r\, dx.$$

(Repeating the same process again).

Repeating the same process again and again, we have

$$= (-1)^r \frac{(2r+1)}{2^{r+1}\, r!} n(n-1)...(n-r+1) \int_{-1}^{+1} x^{n-r} (x^2-1)^r\, dx$$

$$= (-1)^r (-1)^r \frac{(2r+1)\, n!}{2^{r+1} r!\, (n-r)} \int_{-1}^{+1} x^{n-r} (1-x^2)^r\, dx. \qquad \text{...(3)}$$

But $r$ is one of the integers $n, n-2, n-4$... Therefore $n-r$ is one of the numbers 0, 2, 4, 6,......etc., *i.e.* $(n-r)$ is an even integer including zero. So $x^{n-r}(1-x^2)^r$ is an even function of $x$.

Hence from (3), we get

$$a_r=\frac{(2r+1)}{2^{r+1}\,r\,!}\frac{n\,!}{(n-r)\,!}\,2\int_0^1 x^{n-r}\,(1-x^2)^r\,dx$$

$$=\frac{(2r+1)}{2^{r+1}.r\,!}\frac{n\,!}{(n-r)\,!}\int_0^1 t^{(n-r-1)/2}\,(1-t)^r\,dt$$

Putting $x^2=t$ so that $2x\,dx=dt$

$$=\frac{(2r+1)}{2^{r+1}\,r\,!}\frac{n\,!}{(n-r)\,!}\int_0^1 t^{(n-r+1)/2-1}\,(1-t)^{(r+1)-1}\,dt$$

$$=\frac{(2r+1)}{2^{r+1}.r\,!}.\frac{n\,!}{(n-r)\,!}.\frac{\Gamma\left(\frac{n-r+1}{2}\right)\Gamma\,(r+1)}{\Gamma\left(\frac{n-r+1}{2}+r+1\right)}$$

since from Integral Calculus.

$$B\,(l,\,m)=\int_0^1 x^{l-1}\,(1-x)^{m-1}\,dx=\frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)}$$

$$=\frac{(2r+1)}{2^{r+1}.r\,!}.\frac{n\,!}{(n-r)\,!}.\frac{\Gamma\left(\frac{n-r+1}{2}\right)r\,!}{\Gamma\left(\frac{n+r+3}{2}\right)}$$

$$=\frac{(2r+1)}{2^{r+1}}.\frac{n\,!}{(n-r)\,!}$$

$$\times\frac{\Gamma\left(\frac{n-r+1}{2}\right)}{\left(\frac{n+r+1}{2}\right)\left(\frac{n+r-1}{2}\right)\left(\frac{n+r-3}{2}\right)\cdots\left(\frac{n-r+3}{2}\right)\left(\frac{n-r+1}{2}\right)\Gamma\left(\frac{n-r+1}{2}\right)}$$

$$=(2r+1).\frac{n\,(n-1)\,(n-2)\ldots(n-r+1)\,(n-r)\,!}{(n-r)\,!\,(n+r+1)\,(n+r-1)\,(n+r-3)\ldots(n-r+3)\,(n-r+1)}$$

$$\therefore\quad \mathbf{a}_r=(2r+1)\,\frac{n\,(n-1)\,(n-2)\ldots(n-r+2)}{(n+r+1)\,(n+r-1)\ldots(n-r+3)}\qquad\ldots(4)$$

Putting $r=n,\,(n-2),\,(n-4),\ldots\ldots,$ in (4), we have

$$a_n=(2n+1)\,\frac{n\,(n-1)\ldots2}{(2n+1)\,(2n-1)\ldots3}$$

*i.e.,*

$$a_n=\frac{2.3.4\ldots n}{3.5\ldots(2n+1)}.\,(2n+1)$$

$$a_{n-2}=\frac{4.5\ldots n}{5.7\ldots(2n-1)}\,(2n-3)$$

$$=\frac{2.3.4\ldots n}{3.5.7\ldots(2n+1)}.\,\frac{(2n+1)}{2}\,(2n-3)$$

$$a_{n-6}=\frac{2.3\ldots n}{3.5.7\ldots(2n+1)}\cdot\frac{(2n-1)(2n+1)}{2.4}(2n-7) \text{ etc.}$$

Substituting these is (1), we have

$$x^n=\frac{n!}{3.5\ldots(2n+1)}\left[(2n+1)P_n(x)+(2n-3)\frac{(2n+1)}{2}P_{n-2}(x)\right.$$

$$+(2n-7)\frac{(2n+1)(2n-1)}{2.4}P_{n-4}(x)+\ldots$$

$$\left.+\frac{1}{n+1}P_0(x) \text{ or } \frac{3}{n+2}P_1(x)\right] \quad \ldots(5)$$

according as n is even or odd

Since the coefficient is $a_0$ or $a_1$ according as $n$ is even or odd, which from (2) is calculated as follows :

$$a_0=\frac{1}{2}\int_{-1}^{+1} x^n.P_0(x)\,dx=\frac{1}{n+1} \text{ as } P_0(x)=1$$

and $$a_1=\frac{2}{3}\int_{-1}^{+1} x^n P_1(x)\,dx$$

$$=\frac{3}{2}\int_{-1}^{+1} x^n.x\,dx=\frac{3}{n+2}$$

$$x^n=\frac{n!}{2^n}\sum_{k=0}^{(n/2)}\frac{(2n-4k+1)P_{n-2k}(x)}{k!\,(3/2)_{n-k}}.$$ [Raj. 84; Kanpur 84]

where $$\left(\frac{n}{2}\right)=\begin{cases}\dfrac{n}{2} & \text{if } n \text{ is even}\\[2mm] \dfrac{n-1}{2} & \text{if } n \text{ is odd}\end{cases}$$

and $(\alpha)_n=\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)$

$(\alpha)_0=1.$

**Note.** It is suggested to determine the value of $a_0$ and $a_1$ directly from (2) and not by the formula (4).

**§ 2·15. General Results.**

**1.** Let $f$ denote a sectionally continuous function on the interval $(-1, 1)$. Then at each interior point $x$ of the interval at which $f$ is continuous and has derivatives from the right and left Legendre's series corresponding to $f$ converges to $f(x)$, that is

$$f(x)=\sum_{n=0}^{\infty} A_n P_n(x) \qquad (-1<x<1)$$

where the coefficient $A_n$ given by

$$A_n=\frac{2n+1}{2}\int_{-1}^{+1} f(x)\,P_n(x)\,dx$$ [Meerut 82(P)]

is even when $n$ is even, and odd when $n$ is odd.

Hence
$$A_{2n+1}=0$$
and
$$A_{2n}=(4n+1)\int_0^1 f(x)\,P_{2n}(x)\,dx \qquad (n=0,\ 1,\ 2,\ldots)$$

**2.** When $f$ is sectionally continuous on the interval $(0, 1)$ and its one side derivative exists at a point $x$ $(0<x<1)$ at which $f$ is continuous and when

$$A_{2n}=(4n+1)\int_0^1 f(x)\,P_{2n}(x)\,dx.$$

then
$$f(x)=\sum_{n=1}^{\infty} A_{2n}\,P_{2n}\,(x). \qquad (0<x<1)$$

**3.** Similarly when $f$ satisfies the same conditions on the interval $(0, 1)$ and at the point $x$ and when

$$A_{2n+1}=(4n+3)\int_0^1 f(x)\,P_{2n+1}\,(x)\,dx\ (n=0,\ 1,\ 2,\ldots)$$

then
$$f(x)=\sum_{n=1}^{\infty} A_{2n+1}\,P_{2n+1}\,(x).$$

**4.** If $f(x)=c_0+c_1x+c_2x^2+\ldots+c_nx^n+c_{n+1}x^{n+1}+c_{n+2}x^{n+2}+\ldots$ ...(1)

where $c$'s are constants.

If $f(x)$ is also expressed as $f(x)=\sum_{n=0}^{\infty} b_n\,P_n$ ...(2)

then
$$b_n=\frac{1.2.3\ldots n}{1.3.5\ldots(2n-1)}\left[c_n+\frac{(n+1)\,(n+2)}{2\,(2n+3)}\,c_{n+2}\right.$$
$$\left.+\frac{(n+1)\,(n+2)\,(n+3)\,(n+4)}{2.4\,(2n+3)\,(2n+5)}\,c_{n+4}+\ldots\right].$$

**Proof.** Replacing every power of $x$ in (1), by its expansion in $P$'s let us collect the terms which involve $P_n(x)$.

It is clear that when $x^n$, $x^{n+2}$, $x^{n+4}$,..., are expanded in $P$'s each of them will give a term involving $P_n(x)$ and no power of $x$ less than the $n$th can give any term involving $P_n(x)$.

Therefore $c_nx^n$, $c_{n+2}x^{n+2}$, $c_{n+4}x^{n+4}$, in (1) will give each a term involving $P_n(x)$.

Using § 2·14 page 35, we have

$$c_nx^n=c_n\,\frac{n\,!}{1.3.5\ldots(2n+1)}\Big[(2n+1)\,P_n(x)+\ldots\Big]$$

$$c_{n+2}x^{n+2}=c_{n+2}\,\frac{(n+2)\,!}{1.3\ldots(2n+5)}$$

$$\times\left[(2n+5)\,P_{n+2}+(2n+1)\,\frac{(2n+5)}{5}\,P_n+\ldots\right]$$

$$c_{n+4}x^{n+4}=c_{n+4}\,\frac{(n+4)}{1.3\ldots(2n+9)}\left[(2n+9)\,P_{n+4}\right.$$

$$\left.+(2n+5)\,\frac{(2n+9)}{2}\,P_{n+2}+(2n+1).\frac{(2n+9)\,(2n+7)}{2.4}\,P_n+\ldots\right].$$

etc.

Substituting in (1), the coefficient of $P_n\,(x)$ in the expansion of $f(x)$ is given by

$$b_n=\frac{n\,!}{1.3.5\ldots(2n-1)}\left[C_n+\frac{(n+1)\,(n+2)}{2\,(2n+3)}\,C_{n+2}\right.$$

$$\left.+\frac{(n+1)\,(n+2)\,(n+3)\,(n+4)}{2.4\,(2n+3)\,(2n+4)}\,C_{n+4}+\ldots\right].$$

**Proved.**

**§ 2·16. An Important Case.**

To show that

$$\frac{1}{y-x}=\sum_{n=0}^{\infty}(2n+1)\,P_n(x)\,Q_n(x).$$

[Jodhpur 83]

**Proof.** Let $f(x)=\dfrac{1}{y-x}=\dfrac{1}{y\,\{1-x/y\}}=\dfrac{1}{y}\left\{1-\dfrac{x}{y}\right\}^{-1}$

$$=\frac{1}{y}\left\{1+\frac{x}{y}+\frac{x^2}{y^2}+\frac{x^3}{y^3}+\frac{x^4}{y^4}+\ldots+\frac{x^n}{y^n}\right.$$

$$\left.+\frac{x^{n+1}}{y^{n+1}}+\ldots\right\}$$

$$=y^{-1}+xy^{-2}+y^{-3}x^2+y^{-4}x^3+\ldots+y^{-(n+1)}\,x^n$$

$$+y^{-(n+2)}x^{n+1}+\ldots \qquad \ldots(1)$$

As in § 2·15 in (4), if we expand $f(x)$ as

$$f(x)=c_0+c_1x+c_2x^2+\ldots+c_nx^n+c_{n+1}x^{n+1}+\ldots \qquad \ldots(2)$$

and also express as $f(x)=\sum_{0}^{\infty} b_nP_n$

where
$$b_n=\frac{n\,!}{1.3.5\ldots(2n-1)}\left[c_n+\frac{(n+1)\,(n+2)}{2.(2n+3)}\,c_{n+2}\right.$$

$$\left.+\frac{(n+1)\,(n+2)\,(n+3)\,(n+4)}{2.4.(2n+3)\,(2n+5)}\,c_{n+4}+\ldots\right]$$

then comparing (1) and (2), we have

$$c_0=y^{-1},\ c_1=y^{-2},\ c_2=y^{-3},\ldots,\ c_n=y^{-n-1},\ c_{n+1}=y^{-n-2}\text{ etc.}$$

$$\therefore\ b_n=\frac{n\,!}{1.3\ldots(2n-1)}\left[y^{-n-1}+\frac{(n+1)\,(n+2)}{2.(2n+3)}\,y^{-n-3}+\ldots\right]$$

$$=(2n+1)\,Q_n(y) \qquad \text{from §2·14}$$

$$\therefore \quad f(x)=\frac{1}{y-x}\sum_{n=0}^{\infty}(2n+1)\,Q_n(y)\,P_n(x).$$

**Ex. 23.** *Prove that, for all x,*

(a) $x^4=8/35\,P_4(x)+4/7\,P_2(x)+1/5\,P_0(x),$

*and* (b) $x^5=8/63\,P_5(x)+4/9\,P_3(x)+3/7\,P_1(x).$

**Proof.** We know from § 2·23, that

$$x^n=\frac{n!}{3.5\ldots(2n+1)}\left[(2n+1)\,P_n(x)+\frac{(2n-3)}{3}(2n+1)\,P_{n-2}(x)\right.$$
$$\left.+(2n-7)\frac{(2n+1)(2n-1)}{2.4}P_{n-4}(x)+\ldots\right].$$

Hence $n=4$.

$$\therefore \quad x^4=\frac{4!}{3.5.7.9}\left[9.P_4+\frac{59}{2}.P_2(x)+1.\frac{9.7}{2.4}P_0(x)\right]$$
$$=8/35\,P_4(x)+4/7\,P_2(x)+1/5\,P_0(x).$$

(b) Here $n=5$

$$\therefore \quad x^5=\frac{4!}{3.5.7\,9.11}\left[11.P_5(x)+7.\frac{11}{2}P_3(x)+3.\frac{11.9}{2.4}P_1(x)\right]$$
$$=8/63\,P_5(x)+4/9\,P_3(x)+3/7\,P_1(x).$$

**Ex. 24.** *Obtain the first few terms in the representation of the function $f(x)=x$ over the interval $0\leqslant x<1$ in series of Legendre polynomials of even degree to show that*

$x=\frac{1}{2}\,P_0(x)+5/8\,P_2(x)-3/16\,P_4(x)+\ldots(0\leqslant x<1).$

**Proof.** We know that

$$f(x)=\sum_{n=0}^{\infty}A_{2n}\,P_{2n}(x)$$
$$=A_0\,P_0(x)+A_2P_2(x)+A_4P_4(x)+\ldots$$

where $\quad A_{2n}=(4n+1)\int_0^1 f(x)\,P_{2r}(x)\,dx$

for $\quad f(x)=x$

$$A_0=1.\int_0^1 x.P_0(x)\,dx=\int_0^1 x.1\,dx=\tfrac{1}{2}$$

$$A_2=5.\int_0^1 .e^xP_2(x)\,dx$$
$$=5.\int_0^1 x.\frac{1}{2^2.2!}\cdot\frac{d^2}{dx^2}.(x^2-1)^2\,dx.$$

Since by Rodrigue's formula

$$P_n(x)=\frac{1}{2^n.n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n$$

$$=\frac{5}{8}\int_0^1 x.\frac{d^2}{dx^2}(x^2-1)^2\,dx$$

Integrating by parts taking $x$ as first function

$$=\frac{5}{8}\left[\left\{x.\frac{d}{dx}(x^2-1)^2\right\}_0^1-\int_0^1 1.\frac{d}{dx}(x^2-1)^2\,dx\right]$$

$$=\frac{5}{8}\left[0-\left\{(x^2-1)^2\right\}_0^1\right]=\frac{5}{8}$$

and $\quad A_4=9\int_0^1 x.P_4(x)\,dx$

$$=9\int_0^1 x.\frac{1}{2^4.4\,!}\cdot\frac{d^4}{dx^4}(x^2-1)^4\,dx$$

$$=\frac{9}{2^4.4\,!}\int_0^1 x.\frac{d^4}{dx^4}(x^2-1)^4\,dx$$

$$=\frac{3}{2^4.8}\left[\left\{x.\frac{d^3}{dx^3}(x^2-1)^4\right\}_0^1-\int_0^1 1.\frac{d^3}{dx^3}(x^2-1)^4\,dx\right]$$

$$=\frac{3}{16.8}\left[0-\left(\frac{d^2}{dx^2}(x^2-1)^4\right)_0^1\right]$$

$$=\frac{1}{16.8}\left[-8\,(x^2-1)^2\,(7x^2-1)\right]_0^1$$

$$=\frac{1}{16.8}\,[-8]=-\frac{3}{16}\text{ etc.}$$

$f(x)=x=1/2P_0(x)+5/8P_2(x)-3/16P_4(x)+\ldots$ **Proved.**

**Associated Legendre Functions.**

**§ 2·17. Associated Legendre Equation.** **[Jodhpur 81, 84]**

The differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left[n\,(n+1)-\frac{m^2}{1-x^2}\right\}y=0$$

is known as the **Associated Legendre equation.**

**§ 2·18. If v is a solution of Legendre equation.**

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)\,y=0$$

**then** $$(1-x^2)^{m/2}\frac{d^m v}{dx^m}$$

**is a solution of associated Legendre equation.**

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left\{n\,(n+1)-\frac{m^2}{1-x^2}\right\}y=0.$$

**Proof.** Since $v$ is the solution of Legendre's equation, we have

$$(1-x^2)\frac{d^2v}{dx^2}-2x\frac{dv}{dx}+n\,(n+1)\,v=0 \qquad ..(i)$$

Differeniating (i) $m$ times w.r.t. '$x$' by Leibnitz theorem we have

$$\left\{(1-x^2)\frac{d^{m+2}v}{dx^{m+2}}-m.2x\frac{d^{m+1}v}{dx^{m+1}}-\frac{m(m-1)}{2!}.2.\frac{d^m v}{dx^m}\right\}$$
$$-2\left\{x.\frac{d^{m+1}v}{dx^{m+1}}+m.\frac{d^m v}{dx^m}\right\}+n(n+1)\frac{d^m v}{dx^m}=0$$

or $(1-x^2)\dfrac{d^{m+2}v}{dx^{m+2}}-2(m+1).x\dfrac{d^{m+1}v}{dx^{m+1}}+[n(n+1)-m(m+1)]\dfrac{d^m v}{dx^m}=0$

Writing $v_1$ for $\dfrac{d^m v}{dx^m}$, we have

$$(1-x^2)\frac{d^2v_1}{dx^2}-2(m+1)x\frac{dv_1}{dx}+\{n(n+1)-m(m+1)\}v_1=0 \quad \text{...(ii)}$$

Now let $z=(1-x^2)^{m/2}\dfrac{d^m y}{dx^m}=(1-x^2)^{m/2}v_1$

so that $\qquad v_1=(1-x^2)^{-m/2}z$

$$\frac{dv_1}{dx}=(1-x^2)^{-m/2}\frac{dz}{dx}+m(1-x^2)^{-(m/2)-1}.xz$$

and $\dfrac{d^2v_1}{dx^2}=(1-x^2)^{-m/2}\dfrac{d^2z}{dx^2}+2m(1-x^2)^{-(m/2)-1}x\dfrac{dz}{dx}$

$$+mz(1-x^2)^{-(m/2)-1}+m(m+2)x^2z(1-x^2)^{-(m/2)-2}$$

Putting in equation (ii), we have

$$(1-x^2)\left\{(1-x^2)^{-m/2}\frac{d^2z}{dx^2}+2m(1-x^2)^{-(m/2)-1}x\frac{dz}{dx}\right.$$
$$\left.+mz(1-x^2)^{-(m/2)-1}+m(m+2)x^2z(1-x^2)^{-(m/2)-2}\right\}$$
$$-2(m+1)x\left\{(1-x^2)^{-m/2}\frac{dz}{dx}+mxz(1-x^2)^{-(m/2)-1}\right\}$$
$$+\{n(n+1)-m(m+1)\}(1-x^2)^{-(m/2)}z=0.$$

or $(1-x^2)^{-m/2}\left[(1-x^2)\dfrac{d^2z}{dx^2}-2x\dfrac{dz}{dx}+\left\{n(n+1)-\dfrac{m^2}{1-x^2}\right\}z\right]=0$

or $(1-x^2)\dfrac{d^2z}{dx^2}-2x\dfrac{dz}{dx}+\left\{n(n+1)-\dfrac{m^2}{1-x^2}\right\}z=0.$

Hence $z=(1-x^2)^{m/2}\dfrac{d^m v}{dx^m}$ is the solution of associated Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left\{n(n+1)-\frac{m^2}{1-x^2}\right\}y=0.$$

**Proved.**

**§ 2·19. Associated Legendre Function.** **[Poona 70]**

The associated Legendre function $P_n^m(x)$ is defined by

$$P_n^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \qquad \ldots\text{(i)}$$

$m \geqslant 0.$

$P^m{}_n(x)$ thus defined satisfy the associated Legendre equation (see § 2·18).

Using Rodrigue's formula, we have

$$P^m{}_n(x)=(1-x^2)^{m/2}\frac{1}{2^n\,n\,!}\cdot\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n. \qquad \ldots\text{(ii)}$$

Here right hand side is well defined for negative values of $m$ s.t. $m+n \geqslant 0$ *i.e.* $m \geqslant -n$.

Thus form (i) is used to define $P^m{}_n(x)$ for $m \geqslant 0$ while the form (ii) may be used to define $P^m{}_n(x)$ for values of

**§ 2·20. Properties of the associated Legendre functions.**

(i) $\qquad P^0{}_n(x)=P_n(x)$

and (ii) $\qquad P^m{}_n(x)=0 \quad$ if $m>n$.

**Proof.** (i) We have $P^m{}_n(x)=(1-x^2)^{m/2}\dfrac{d^m}{dx^m}P_n(x)$.

Taking $m=0$, we have

$$P^0{}_n(x)=P_n(x)$$

**Proved.**

(ii) We have $P^m{}_n(x)=(1-x^2)^{m/2}\dfrac{d^m}{dx^m}P_n(x)$.

$P_n(x)$ is a polynomial of degree $n$. If $m>n$, then

$$\frac{d^m}{dx^m}P_n(x)=0.$$

Hence from ($a$), $P^m{}_n(x)=0 \quad$ if $m>n$. **Proved.**

**§ 2·21. Orthogonal Properties of associated Legendre's functions.**

**To prove that**

$$\int_{-1}^{+1} P^m{}_n(x)\,P^m{}_{n'}(x)\,dx=\frac{2(n+m)\,!}{(2n+1)(n-m)}\,\delta_{nn'}$$

**where $\delta_{nn'}$ is Kronecker delta.** **[Meerut 83]**

**Proof.**

**Case I.** When $n \neq n'$.

$P^m{}_n(x)$ and $P^m{}_{n'}(x)$ satisfy the corresponding associated Legendre equation.

Now associated Legendre equation may be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+\left\{n(n+1)-\frac{m^2}{1-x^2}\right\}y=0$$

$$\therefore \quad \frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P^m{}_n(x)\right\}+\left\{n(n+1)-\frac{m^2}{1-x^2}\right\}P^m{}_n(x)=0 \quad \ldots(i)$$

and $$\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P^m{}_{n'}(x)\right\}+\left\{n'(n'+1)-\frac{m^2}{1-x^2}\right\}P^m{}_{n'}(x)=0 \quad \ldots(ii)$$

Multiplying (i) by $P^m{}_{n'}(x)$ and (ii) by $P^m{}_n(x)$ and then subtracting, we have

$$P^m{}_{n'}(x).\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P^m{}_n(x)\right\}$$
$$-P^m{}_n(x).\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P^m{}_{n'}(x)\right\}$$
$$+\left\{n(n+1)-n'(n'+1)\right\}P^m{}_n(x)\,P^m{}_{n'}(x)=0.$$

Integrating between the limits $-1$ to $1$ and proceeding similarly as in § 2·7 (i), we have

$$\{n(n+1)-n'(n'+1)\}\int_{-1}^{+1}P^m{}_n(x)\,P^m{}_{n'}(x)\,dx=0$$

$$\therefore \quad \int_{-1}^{+1}P^m{}_n(x)\,P^m{}_{n'}(x)\,dx=0, \text{ since } n\neq n'.$$

**Case II.** When $n=n'$.

Since $P^m{}_n(x)=(1-x^2)^{m/2}\dfrac{d^m}{dx^m}P_n(x)$ if $m>0$.

$$\therefore \quad \int_{-1}^{+1}P^m{}_n(x).P^m{}_{n'}(x)\,dx=\int_{-1}^{+1}\{P_n{}^m(x)\}^2\,dx$$

$$=\int_{-1}^{+1}(1-x^2)^m\left\{\frac{d^m}{dx^m}P_n(x)\right\}\left\{\frac{d^m}{dx^m}P_n(x)\right\}dx$$

$\left(\text{Integrating by parts taking } \dfrac{d^m}{dx^m}P_n(x) \text{ as the second function}\right)$

$$=\left[\left\{(1-x^2)^m\frac{d^m}{dx^m}P_n(x)\right\}\left\{\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\}\right]_{-1}^{+1}$$
$$-\int_{-1}^{+1}\left\{\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\}\frac{d}{dx}\left\{(1-x^2)^m\frac{d^m}{dx^m}P_n(x)\right\}dx$$
$$=-\int_{-1}^{+1}\left\{\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\}\frac{d}{dx}\left\{(1-x^2)\frac{d^m}{dx^m}P_n(x)\right\}dx. \quad \ldots(iii)$$

Now $P_n(x)$ satisfy Legendre equation

$$\therefore \quad (1-x^2)\frac{d^2P_n(x)}{dx^2}-2x\frac{dP_n(x)}{dx}+n(n+1)\,P_n(x)=0.$$

Differentiating $(m-1)$ times w.r.t. '$x$' by Leibnitz's theorem, we have

$$(1-x^2)\frac{d^{m+1}}{dx^{m+1}}P_n(x)-2mx\frac{d^m}{dx^m}P_n(x)$$
$$+\{n(n+1)-(m-1)m\}\frac{d^{m-1}}{dx^{m-1}}P_n(x)=0.$$

Multiplying by $(1-x^2)^{m-1}$, we have

$$(1-x^2)^m\frac{d^{m+1}}{dx^{m+1}}P_n(x)-2xm(1-x^2)^{m-1}\frac{d^m P_n(x)}{dx^m}$$
$$+\{n(n+1)-(m-1)m\}(1-x^2)^{m-1}.\frac{d^{m-1}P_n(x)}{dx^{m-1}}=0$$

or
$$\frac{d}{dx}\left\{(1-x^2)^m\frac{d^m}{dx^m}P_n(x)\right\}$$
$$=-(n+m)(n-m+1)(1-x^2)^{m-1}.\frac{d^{m-1}}{dx^{m-1}}P_n(x).$$

Putting in (iii), we have

$$\int_{-1}^{+1}\{P^m{}_n(x)\}^2\,dx$$
$$=\int_{-1}^{+1}\left\{\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\}.(n+m)(n-m+1)(1-x^2)^{m-1}$$
$$\times\frac{d^{m-1}}{dx^{m-1}}P_n(x)\,dx$$
$$=(n+m)(n-m+1)\int_{-1}^{+1}(1-x^2)^{m-1}\left\{\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\}^2dx$$
$$=(n+m)(n-m+1)\int_{-1}^{+1}\{P_n{}^{m-1}(x)\}^2\,dx.$$

Repeating the same process again and again $m$ times, we have

$$\int_{-1}^{+1}\{P^m{}_n(x)\}^2\,dx=\{(n+m)(n-m+1)\}\{n+m-1)(n-m+1)\}\ldots$$
$$\ldots\{(n+1)n\}\int_{-1}^{+1}\{P^0{}_n(x)\}^2\,dx$$
$$=(n+m)(n+m-1)\ldots(n+1).n.(n-1)\ldots$$
$$\ldots(n-m+2)(n-m+1)\times\int_{-1}^{+1}\{P_n(x)\}^2\,dx$$
$$\because\quad P^0{}_n(x)=P_n(x)$$

$$=\frac{\{(n+m)\}!}{\{(n-m)\}!}.\frac{2}{2n+1} \qquad \ldots(iv)$$

since $$\int_{-1}^{+1} \{(P_n(x)\}^2\, dx = \frac{2}{2n+1}$$

Now if $m < 0$, say $m = -l$ when $l > 0$,

then $$\int_{-1}^{+1} \{P^m{}_n(x)\}^2\, dx = \int_{-1}^{+1} \{P^{-l}{}_n(x)\}^2\, dx$$

$$= \int_{-1}^{+1} \left\{(-1)^n \frac{\{(n-l)\}\,!}{\{(n+l)\}\,!}\right\}^2 \{P^l{}_n(x)\}^2\, dx$$

Since $P_n^{-m}(x) = (-1)^m \dfrac{\{(n-m)\}\,!}{\{(n+m)\}\,!} P_n^m(x)$

$$= \left[\frac{\{(n-l)\}\,!}{\{(n+l)\}\,!}\right]^2 \int_{-1}^{+1} \{P^l{}_n(x)\}^2\, dx$$

$$= \left[\frac{\{(n-l)\}\,!}{\{(n+l)\}\,!}\right]^2 \cdot \frac{\{(n+l)\}\,!}{\{(n-l)\}\,!} \cdot \frac{2}{2n+1}$$

from the last derived result (iv) for $m > 0$.
Here $l > 0$.

$$= \frac{(n-l)\,!}{(n+1)\,!} \cdot \frac{2}{2n+1} = \frac{(n+m)\,!}{(n-m)\,!} \cdot \frac{2}{2n+1}$$

Hence $$\int_{-1}^{+1} \mathbf{P}^m{}_n(\mathbf{x}) . \mathbf{P}^m{}_{n'}(\mathbf{x})\, \mathbf{dx} = \frac{2\,(\mathbf{n}+\mathbf{m})}{(2\mathbf{n}+1)\,(\mathbf{n}-\mathbf{m})\,!} . \delta_{mn} .$$

**Proved.**

**§ 2·22. Recurrence Formula for Associated Legendres Function.**

**(I)** $$\mathbf{P}_n{}^{m+1}(\mathbf{x}) - \frac{2\mathbf{mx}}{\sqrt{\{(1-\mathbf{x}^2)\}}} \mathbf{P}_n{}^m(\mathbf{x}) + \{\mathbf{n}\,(\mathbf{n}+1) - \mathbf{m}\,(\mathbf{m}-1)\}\, \mathbf{P}_n{}^{m-1}(\mathbf{x})\} = 0.$$

**Proof.** Since $P_n(x)$ is the solution of Legendre's equation.

$$\therefore \quad (1-x^2) \frac{d^2}{dx^2} P_n(x) - 2x \frac{d}{dx} P_n(x) + n\,(n+1)\, P_n(x) = 0.$$

Differentiating $(m-1)$ times, w.r.t. '$x$' by Leibnitz's theorem, we have

$$\left\{(1-x^2) \frac{d^{m+1}}{dx^{m+1}} P_n(x) - 2x\,(m-1) \frac{d^m}{dx^m} P_n(x) - 2 . \frac{(m-1)\,(m-2)}{2\,!} . \frac{d^{m-1}}{dx^{m-1}} P_n(x)\right\}$$

$$-2 \left\{x . \frac{d^m}{dx^m} P_n(x) + (m-1).1. \frac{d^{m-1}}{dx^{m-1}} P_n(x)\right.$$

$$+ n\,(n+1) \frac{d^{m-1}}{dx^{m-1}} P_n(x) = 0$$

or
$$(1-x^2)\frac{d^{m+1}}{dx^{m+1}}P_n(x)-2xm\frac{d^m}{dx^m}P_n(x)$$
$$+\{n(n+1)-(m-1)m\}\frac{d^{m-1}}{dx^{m-1}}P_n(x)=0.$$

Multiplying by $(1-x^2)^{(m-1)/2}$, we have
$$(1-x^2)^{(m+1)/2}\frac{d^{m+1}}{dx^{m+1}}P_n(x)-2xm(1-x^2)^{-1/2}.(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$$
$$+\{n(n+1)-(m-1)m\}(1-x^2)^{(m-1)/2}\frac{d^{m-1}}{dx^{m-1}}P_n(x)=0$$

or
$$P_n^{m+1}(x)-\frac{2mx}{\sqrt{(1-x^2)}}P^m{}_n(x)+\{n(n+1)$$
$$-(m-1)m\}P_n^{m-1}(x)=0.$$
$$\left\{\text{Since } P^m{}_n(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)\right\}$$
**Proved.**

**Note (I).** Recurrence relation is relationship between associated Legendre function with the same value $n$ and consecutive $m$ values.

**(II)** $(2n+1)\,xP^m{}_n(x)=(n+m)\,P^m{}_{n-1}(x)$
$-(n-m+1)\,P^m{}_{n+1}(x).$

**Proof.** From recurrence formula (I) and (III) for Legendre's polynomial, we have
$$(2n+1)\,xP_n(x)=(n+1)\,P_{n+1}(x)+nP_{n-1}(x) \quad \text{...(i)}$$
and
$$(2n+1)\,P_n(x)=\frac{d}{dx}P_{n+1}(x)-\frac{d}{dx}P_{n-1}(x). \quad \text{...(ii)}$$

Differentiating (i) $m$ times w.r.t. '$x$' by Leibnitz's theorem, we have
$$(2n+1)\left\{x\frac{d^m}{dx^m}P_n(x)+m.1.\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\}$$
$$=(n+1)\frac{d^m}{dx^m}P_{n+1}(x)+n\frac{d^m}{dx^m}P_{n-1}(x) \quad \text{...(iii)}$$

Differentiating (ii) $(m-1)$ times w.r.t. '$x$' by Leibnitz's theorem, we have
$$(2n+1)\frac{d^{m-1}}{dx^{m-1}}P_n(x)=\frac{d^m}{dx^m}P_{n+1}(x)-\frac{d^m}{dx^m}P_{n-1}(x). \quad \text{...(iv)}$$

Substituting the value of $(2n+1)\frac{d^{m-1}}{dx^{m-1}}P_n(x)$ from (iii) in (ii) we have
$$(2n+1)\,x\frac{d^m}{dx^m}P_n(x)+m\left\{\frac{d^m}{dx^m}P_{n+1}(x)-\frac{d^m}{dx^m}P_{n-1}(x)\right\}$$

$$=(n+1)\frac{d^m}{dx^m}P_{n+1}(x)+n\frac{d^m}{dx^m}P_{n-1}(x)$$

or
$$(2n+1)\,x\frac{d^m}{dx^m}P_n(x)=(n+m)\frac{d^m}{dx^m}P_{n-1}(x)$$
$$+(n-m+1)\frac{d^m}{dx^m}P_{n+1}(x).$$

Multiplying by $(1-x^2)^{m/2}$, we have

$$(2n+1)\,x.(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)=(n+m)(1-x^2)^{m/2}\frac{d^m}{dx^m}P_{n-1}(x)$$
$$+(n-m+1)(1-x^2)^{m/2}\frac{d^m}{dx^m}P_{n+1}(x)$$

or $\quad (2n+1)\,xP^m{}_n(x)=(n+m)\,P^m{}_{n-1}(x)+(n-m+1)\,P^m{}_{n+1}(x).$

$$\left\{\text{Since } P^m{}_n(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)\right\}.$$

**Proved.**

**Note.** This recurrence formula is the relationship between associated Legendre functions with same value $m$ and consecutive $n$ values.

**(III)** $\sqrt{\{(1-x^2)\}}\,P^m{}_n(x)=\dfrac{1}{(2n+1)}\left\{P^{m+1}_{n+1}(x)-P^{m+1}_{n-1}(x)\right\}$

**Proof.** Differentiating recurrence formula (III) for Legendre's polynomial, $m$ times w.r.t. '$x$', we have

$$(2n+1)\frac{d^m}{dx^m}P_n(x)=\frac{d^{m+1}}{dx^{m+1}}P_{n+1}(x)-\frac{d^{m+1}}{dx^{m+1}}P_{n-1}(x).$$

Multiplying by $(1-x^2)^{(m+1)/2}$, we have

$$(2n+1)(1-x^2)^{1/2}.(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$$
$$=(1-x^2)^{(m+1)/2}.\frac{d^{m+1}}{dx^{m+1}}P_{n+1}(x)-(1-x^2)^{(m+1)/2}.\frac{d^{m+1}}{dx^{m+1}}P_{n-1}(x)$$

or
$$(2n+1)\sqrt{\{(1-x^2)\}}\,P^m_n(x)=P^{m+1}_{n+1}(x)-P^{m+1}_{n-1}(x)$$

or
$$\sqrt{\{(1-x^2)\}}\,P^m{}_n(x)=\frac{1}{(2n+1)}\left\{P^{m+1}_{n+1}(x)-P^{m+1}_{n-1}(x)\right\}$$

**Proved.**

**(IV).** $\sqrt{\{(1-x^2)\}}\,P^m{}_n(x)+\dfrac{1}{(2n+1)}\Big\{(n+m)(n+m-1)\,P^{m-1}_{n-1}(x)$
$$-(n-m+1)(n-m-2)\,P^{m-1}_{n+1}(x)\Big\}.$$

**Proof.** From recurrence formula II, we have

$$P^m{}_n(x)=\frac{1}{(2n+1)}\{(n+m)\,P^m{}_{n-1}(x)+(n-m+1)\,P^m{}_{n+1}(x)\}$$

Putting this value of $xP^m_n(x)$ in recurrence formula I, we have

$$P_n^{m+1}(x)-\frac{2m}{\sqrt{\{(1-x^2)\}}}\frac{1}{(2n+1)}\{(n+m)\ P^m_{n-1}(x)$$
$$+(n-m+1)\ P^m_{n+1}(x)\}$$
$$+\{n\ (n+1)-m\ (m-1)\}\ P_n^{m-1}(x)=0. \quad \ldots(i)$$

Now in recurrence formula III replacing $m$ by $(m-1)$, we have

$$\sqrt{\{(1-x^2)\}}\ P_n^{m-1}(x)=\frac{1}{(2n+1)}\ \{(P^m_{n+1}(x)-P^n_{n-1}(x)\} \quad \ldots(ii)$$

Putting the value of $P_n^{m-1}(x)$ from (ii) in (i) we have

$$P_n^{m+1}(x)-\frac{2m}{(2n+1)\ \sqrt{\{(1-x^2)\}}}\ \{(n+m)\ P^m_{n-1}(x)$$
$$+(n-m-1)\ P^m_{n+1}\ (x)\}+\{n\ (n+1)-m\ (m-1)\}$$
$$\times\frac{1}{(2n+1)\sqrt{(1-x^2)}}\ \{P^m_{n+1}(x)-P^m_{n-1}(x)\}=0$$

or $$\sqrt{(1-x^2)}\ P_n^{m+1}(x)=\frac{1}{(2n+1)}\ [\{2m\ (n+m)+n\ (n+1)$$
$$-m\ (m-1)\}\ P^m_{n-1}(x)+\{2m\ (n-m+1)-n\ (n+1)$$
$$-m\ (m-1)\}\ P^m_{n+1}(x)]$$

or $$\sqrt{\{(1-x^2)\}}\ P_n^{m+1}(x)=\frac{1}{(2n+1)}\ [(n+m)\ (n+m+1)\ P^m_{n-1}(x)$$
$$-(n-m)\ (n-m+\ )\ P^m_{n+1}(x)]$$

Writing $(m-1)$ for $m$, we have

$$\sqrt{\{(1-x^2)\}}\ P^m_n(x)=\frac{1}{(2n+1)}\ [(n+m-1)\ (n+m)\ P^{m-1}_{n-1}\ (x)$$
$$-(n-m+1)\ (n-m+2)\ P^{m-1}_{n+1}\ (x)$$

**Proved.**

**Ex. 25.** *Prove that*

$$\int_0^1 x^m P_n(x)\ dx=\frac{m\ (m-1)\ (m-2)\ldots(m-n+2)}{(m+n+1)\ (m+n-1)\ldots(m-n+3)}$$

*where $m>n-1$, and $n$ is a positive integer.*

**Sol.** From Rodrigue's Formula, we have

$$P_n\ (x)=\frac{1}{2^n\ n\ !}\frac{d^n}{dx^n}\ (x^2-1)^n$$

$$\therefore\ \int_0^1 x^m\ P_n(x)\ dx=\frac{1}{2^n\ n\ !}\int_0^1 x^m\ \frac{d^n}{dx^n}\ (x^2-1)^n\ dx$$
$$=\frac{1}{2^n\ n\ !}\left[\left\{x^m\ \frac{d^{n-1}}{dx^{n-1}}\ (x^2-1)^n\right\}\right]_0^1$$
$$-m\int_0^1\left[x^{m-1}\ \frac{d^{n-1}}{dx^{n-1}}\ (x^2-1)^n\ dx\right]$$

(Integrating by parts taking $x^m$ as Ist function)

$$=\frac{(-1).m}{2^n\, n\,!}\int_0^1 x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

$$=\frac{(-1)^2 m.(m-1)}{2^n\, n\,!}\int_0^1 x^{m-2}\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\,dx$$

(Again integrating by parts)

Continuing the same process of integration $n$ times, we have

$$\int_0^1 x^m P_n(x)\,dx=\frac{(-1)^n\, m(m-1)\,.\,(m-n+1)}{2^n\, n\,!}\int_0^1 x^{m-n}(x^2-1)^n dx$$

$$=\frac{m(m-1)\ldots(m-n+1)}{2^n\, n\,!}\int_0^1 \tfrac{1}{2}.x^{m-n-1}(1-x^2)^n 2x\,dx$$

Putting $x^2=z$ so that $2x\,dx=dz$

$$=\frac{m\,(m-1)\ldots(m-n+1)}{2^{n+1}\, n\,!}\int_0^1 z^{(m-n-1)/2}\,(1-z)^n\,dz\cdot$$

$$=\frac{m\,(m-1)\ldots(m-n+1)}{2^{n+1}\, n\,!}\,B\left(\frac{m-n-1}{2}+1,\, n+1\right)$$

since $\int_0^1 z^{m-1}\,(1-z)^{n-1}\,dz=B\,(m,\,n)$

$$=\frac{m\,(m-1)\,..(m-n+1)}{2^{n+1}\, n\,!}\,B\left(\frac{m-n+1}{2},\, n+1\right)$$

$$=\frac{m(m-1)\ldots(m-n+1)}{2^{n+1}\, n\,!}\;\frac{\Gamma\left(\frac{m-n+1}{2}\right)\Gamma\,(n+1)}{\Gamma\left(\frac{m-n+1}{2}+n+1\right)}$$

Since $B\,(m,\,n)=\dfrac{\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(m+n)}$

$$=\frac{m\,(m-1)\ldots(m-n+1)}{2^{n+1}\, n\,!}\;\frac{\Gamma\left(\frac{m-n+1}{2}\right)n\,!}{\Gamma\left(\frac{m+n+3}{2}\right)}$$

$$=\frac{m\,(m-1)\ldots(m-n+1)}{2^{n+1}\, n\,!}$$

$$\times\frac{\Gamma\left(\frac{m-n+1}{2}\right)n\,!}{\frac{m+n+1}{2}.\,\frac{m+n-1}{2}\cdots\frac{m-n+1}{2}\,\Gamma\left(\frac{m-n+1}{2}\right)}$$

$$=\frac{m(m-1)\ldots(m-n+1)}{2^{n+1}\,\cancel{n\,!}}\cdot\frac{\cancel{n\,!}\,2^{n+1}}{(m+n+1)(m+n-1)\ldots(m-n+1)}$$

$$=\frac{m(m-1)(m-2)\ldots(m-n+2)(m-n+1)}{(m+n+1)(m+n-1)\ldots(m-n+3)(m-n+1)}$$

$$=\frac{m(m-1)(m-2)\ldots(m-n+2)}{(m+n+1)(m+n-1)\ldots(m-n+3)}$$

**Proved.**

**§ 2·23. Trigonometrical Series for $P_n(x)$.**

**To express $P_n(\cos\theta)$ as a series in cosines of multiples of $\theta$.**

We know that

$$\sum_{n=0}^{\infty} h^n P_n(x)=(1-2xh+h^2)^{-1/2}$$

$$\therefore \quad \sum_{n=0}^{\infty} h^n P_n(\cos\theta)=(1-2h\cos\theta+h^2)^{-1/2}$$

$$=\{1-h(e^{i\theta}+e^{-i\theta})+h^2\}^{-1/2}$$

$$=\{(1-he^{i\theta})(1-he^{-i\theta})\}^{-1/2}$$

$$=(1-he^{i\theta})^{-1/2}(1-he^{-i\theta})^{-1/2}$$

$$=\left[1+\tfrac{1}{2}he^{i\theta}+\frac{1.3}{2.4}h^2e^{2i\theta}+\ldots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}h^ne^{ni\theta}+\ldots\right]$$

$$\times\left[1+\tfrac{1}{2}he^{-i\theta}+\frac{1.3}{2.4}h^2e^{-2i\theta}+\ldots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}h^ne^{-ni\theta}+\ldots\right]$$

Equating the coefficient of $h^n$ from the two sides, we have

$$P_n(\cos\theta)=\frac{1.3\ldots(2n-1)}{2.4\ldots(2n)}(e^{ni\theta}+e^{-ni\theta})+\frac{1.3\ldots(2n-3)}{2.4.6\ldots(2n-2)}\cdot\frac{1}{2}\cdot(e^{(n-2)i\theta}+e^{-(n-2)i\theta})$$

$$+\frac{1.3\ldots(2n-5)}{2.4\ldots(2n-4)}\cdot\frac{1.3}{2.4}(e^{(n-4)i\theta}+e^{-(n-4)i\theta}+\ldots)$$

or $$P_n(\cos\theta)=\frac{1.3\ldots(2n-1)}{2.4\ldots(2n)}\left[2\cos n\theta+\frac{2n}{2n-1}\cdot\frac{1}{2}\,2\cos(n-2)\theta\right.$$

$$\left.+\frac{(2n)(2n-2)}{(2n-1)(2n-3)}\cdot\frac{1.3}{2.4}\,2\cos(n-4)\theta+\ldots\right]$$

Which is the required expression.

**Ex. 26.** *Prove that*

$$\int_0^{\pi} P_n(\cos\theta)\cos n\theta\, d\theta=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}.\pi=B(n+\tfrac{1}{2},\tfrac{1}{2}).$$

**(Raj. 85)**

**Solution.** We know that

$$P_n(\cos\theta)=\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\left[2\cos n\theta+\frac{2n}{2n-1}\cdot\frac{1}{2}.2\cos(n-2)\theta\right.$$

$$\left.+\frac{2n.(2n-2)}{(2n-1)(2n-3)}\,\frac{1.3}{2.4}\,2\cos(n-4)\theta+\ldots\right.$$

$$\therefore \quad \int_0^{\pi} P_n(\cos\theta)\cos n\theta\, d\theta=\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\times$$

$$\left[2\int_0^\pi \cos^2 n\theta+\frac{2n}{(2n-1)}\cdot\frac{1}{2}\cdot 2\int_0^\pi \cos(n-2)\,\theta\cos n\theta\,d\theta\right.$$
$$\left.+\frac{2n\,(2n-2)}{(2n-1)\,(2n-3)}\cdot\frac{1.3}{2.4}\,2\int_0^\pi \cos(n-4)\,\theta\cos n\theta\,d\theta+\ldots\right]$$
$$=\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\int_0^\pi (1+\cos 2n\theta)\,d\theta=\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}.\pi$$

Since all other integrals are zero.

$$=\frac{\frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{3}{2}\frac{1}{2}\sqrt{\pi}.\sqrt{\pi}}{1.2.3\ldots n}$$

$$=\frac{\Gamma\left(\frac{2n+1}{2}\right).\Gamma(\frac{1}{2})}{\Gamma(n+1)}=\frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2}+\frac{1}{2})}=B\,(n+\tfrac{1}{2},\tfrac{1}{2}).$$

Hence the result. **Proved.**

## EXECISE ON CHAPTER II

1. Prove that $P_n'-P'_{n-2}=(2n-1)\,P_{n-1}$. **[Rohilkhand 84]**
 [**Hint.** Replace $n$ by $n-1$ in Recu. formula III]
2. Prove that $xP'_9=P'_8-2P_9$.
 [**Hint.** Put $n=2$ in Rec. formula II].
3. Show that $11\,(x^2-1)\,P'_5=30\,(P_6-P_4)$.
 [**Hint.** Put $n=5$ in Beltrami's Result § 2·9].
4. Prove that, for all $x$
 (*a*) $x^2=1/3\,P_0(x)+2/3\,P_2(x)$. **[Kanpur 84]**
 (*b*) $x^3=3/5\,P_1(x)+2/5\,P_3(x)$.
5. Express $x^7$ as a series in Legendre's polynomials.
 $$\left[\text{Ans. } \frac{16}{529}P_7+\frac{8}{39}P_5+\frac{14}{33}P_3+\frac{1}{3}P_1\right]$$
6. Prove that
 $$\int_0^1 P_{2n}(x)\,P_{2n+1}(x)=\int_{-1}^1 P_{2n}(x)\,P_{2n-1}(x)\,dx.$$
7. Show that $\int_{-1}^1 xP_n(x).P_{n-1}(x)\,dx=\frac{2}{4n^2-1}$
 [**Hint.** Using Rec. formula I.
 $$\int_{-1}^1 xP_n(x).P_{n-1}(x)=\frac{n-1}{2n+1}\int_{-1}^1 P_{n+1}(x)\,P_{n-1}(x)\,dx$$
 $$+\frac{n}{2n+1}\int_{-1}^1 \{P_{n-1}(x)\}^2\,dx$$
 now use § 2·7].

8. Prove that $P_1(x)=\frac{1}{\pi}\int_0^{\pi}\{x+\sqrt{(x^2-1)}\cos\theta\}\,d\theta$.

[Hint. Proceeding as in § 2·6, I equate the coefficient of $h$]

9. Show that $\int_{-1}^{1} f(x).P_m(x)=0$ if $m$ is even and $f$ is odd.

Also show that $\int_{-1}^{1} x^r \cdot P_m(x)\,dx=0$ if $r<m$. What is your opinion for $r\geqslant m$? Give reasons for your answer.

10. What would be the degree of polynomial $f(x)$ so that

$$\int_{-1}^{1} f(x)\,P_n(x)\,dx=0.$$

[Meerut 78]

11. Prove that $\int_{-1}^{+1} P_m(x)\,P_n(x)\,dx=\frac{2}{2n+1},\ \delta_{mn}$ where $\delta_{mn}$ is the kronecker delta, [Meerut 79 (S), 83]

[Hint. See § 2·7]

12. If $f(x)$ is a polynomial of degree $n$, then prove that :

$$f(x)=\sum_{r=0}^{n} C_r\,P_r(x)$$

where $C_r=(r+\frac{1}{2})\int_{-1}^{1} f(x)\,P_r(x)\,dx$. [Meerut 82P]

13. Use Rodrigue's formula $P_n(x)=\frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n$ for Legendre's polynomial $P_n(x)$, show that

$$\int_{-1}^{1} f(x)\,P_n(x)\,dx=\frac{(-1)^n}{2^n\,n!}\int_{-1}^{1} f^n(x).(x^2-1)^n\,dx$$

in which $f$ is a function continuous in $(-1, 1)$, and $f^n(x)$ denotes the $n$th derivative of $f$. Hence deduce that

$$\int_{-1}^{1} x^m\,P_n(x)\,dx=0,\ m<n$$

$$=\frac{2^{n+1}(n!)^2}{(2n+1)!},\ m=n.$$ [G.N.U.A. 81]

14. Prove that $\int_x^1 P_n(x)\,dx=\frac{1}{2n+1}[P_{n-1}(x)-P_{n+1}(x)]$

[Meerut 84 (P)]

[Hint. Integrate Recc. formula III between the limits $x$ to 1].

15. Evaluate $\int_{-1}^{+1} x^4\,P_6(x)\,dx$. [Kanpur 85]

16. Prove that $\int_{-1}^{+1}\frac{P_n(x)}{\sqrt{(1-2xh+h^2)}}\,dx=\frac{2h^n}{2n+1}$. [Meerut 87]

# 3

# Legendre's Function of the Second Kind $Q_n$ (x)

## § 3·1. Legendre's function of the second kind.

In § 2·3, we have defined the Legendre's function of the second kind, $Q_n$, such that

$$Q_n(x) = \frac{n!}{1\cdot 3 \ldots (2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2\cdot(2n+3)}\cdot x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\,(2n+3)(2n+5)}\,x^{-n-5}\ldots\right]$$

or $$Q_n(x) = \frac{2^n\, n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)\,x^{-(n+2r+1)}}{2^r\, r!\,(2n+3)(2n+5)\ldots(2n+2r+1)}.$$

## § 3·2. Neumann's Integral.

*If n is zero or a positive integer,*

$$Q_1(y) = \tfrac{1}{2}\int_{-1}^{+1}\frac{P_n(x)}{y-x}\,dx$$

where $y > 1$ and $-1 \leqslant x \leqslant 1$.

**Proof.** In § 2·16, we have proved that

$$\frac{1}{y-x} = \sum_{n=0}^{\infty}(2n+1)\,Q_n(y)\,P_n(x).$$

Multiplying both sides by $P_n(x)$ and then integrating w.r.t. '$x$' between the limits $-1$ to $+1$, we have

$$\int_{-1}^{+1}\frac{P_n(x)}{y-x}\,dx = Q_n(y)\int_{-1}^{+1}\{P_n(x)\sum_{n=0}^{\infty}(2n+1)\,P_n(x)\}\,dx$$

$$= (2n+1)\,Q_n(y)\int_{-1}^{+1}P_n^2(x)\,dx.$$

Since $\int_{-1}^{+1}P_n(x)\,P_m(x)\,dx = 0$ if $m \neq n$

$$=(2n+1)\,Q_n(y).\frac{2}{2n+1}=2Q_n(y) \qquad \text{from § 2·7}$$

$$\therefore \quad Q_n(y)=\tfrac{1}{2}\int_{-1}^{+1}\frac{P_n(x)}{y-x}\,dx.$$

## § 3·3. Recurrence Formula for $Q_n(x)$.

(I) $Q'_{n+1}(x)-Q'_{n-1}(x)=(2n+1)\,Q_n(x)$.

[Agra 80, 83 ; Rohilkhand 76]

**Proof.** We know that

$$Q_n(x)=\frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\,r!\,(2n+3)(2n+5)\dots(2n+2r+1)}.$$

Differentiating with respect to '$x$' we have

$$Q'_n(x)=-\frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r+1)!\,x^{-(n+2r+2)}}{2^r\,r!\,(2n+3)(2n+5)\dots(2n+2r+1)}$$

Replacing $n$ by $(n-1)$ and $(n+1)$, we have

$$Q'_{n-1}(x)=-\frac{2^{n-1}(n-1)!}{(2n-1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\,r!\,(2n+1)(2n+3)\dots(2n+2r-1)}$$

$$=-\frac{2^n n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\,r!\,(2n+1)(2n+3)\dots(2n+2r-1)}$$

and

$$Q'_{n+1}(x)=-\frac{2^{n+1}(n+1)!}{(2n+3)!}\sum_{r=0}^{\infty}\frac{(n+2r+2)\,x^{-(n+2r+3)}}{2^r\,r!\,(2n+5)(2n+7)\dots(2n+2r+3)}$$

$$=-\frac{2^n n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r+2)!\,x^{-(n+2r+3)}}{2^r\,r!\,(2n+1)(2n+3)\dots(2n+2r+3)}$$

Now $(2n+1)\,Q_n(x)+Q'_{n-1}(x)$

$$=\frac{2^n n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\,r!\,(2n+3)(2n+5)\dots(2n+2r+1)}$$

$$-\frac{2^n n!}{2n!}\sum_{r=0}^{\infty}\frac{(2+2r)!\,x^{-(n+2r+1)}}{2^r\,r!\,(2n+1)(2n+3)\dots(2n+2r-1)}$$

$$=\frac{2^n n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,\{(2n+1)-(2n+2r+1)\}\,x^{-(n+2r+1)}}{2^r\,r!\,(2n+1)(2n+3)\dots(2n+2r+1)}$$

$$=\frac{2^n n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,(-2r)\,x^{-(n+2r+1)}}{2^n\,r!\,(2n+1)(2n+3)\dots(2n+2r+1)}$$

$$=\frac{-2^n n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^{r-1}(r-1)!\,(2n+1)(2n+3)\ldots(2n+2r+1)}$$

(Since term corresponding to $r=0$ is zero)

$$=-\frac{2^n n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2s+2)!\,x^{-(n+2r+3)}}{2^s s!\,(2n+1)(2n+3)\ldots(2n+2s+3)}$$

Putting $r-1=s$

$$=Q'_{n+1}(x).$$

**Hence $Q'_{n+1}(x)-Q'_{n-1}(x)=(2n+1)\,Q_n(x)$.** **Proved.**

**(II) $Q'_{n+1}(x)-xQ'_n(x)=(n+1)\,Q_n(x)$.**

**Proof.** We know that

$$Q_n(x)=\frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+3)(2n+5)\ldots(2n+2r+1)}$$

As in (I), we have

$$Q'_n(x)=-\frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r+1)!\,x^{-(n+2r+2)}}{2^r r!\,(2n+3)(2n+5)\ldots(2n+2r+1)}$$

and $$Q'_{n+1}(x)=-\frac{2^n n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r+2)!\,x^{-(n+2r+3)}}{2^r r!\,(2n+1)(2n+3)\ldots(2n+2r+3)}$$

$\therefore\quad (n+1)\,Q_n(x)+xQ'_n(x)$

$$=\frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+1)(n+2r)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+3)(2n+5)\ldots(2n+2r+1)}$$

$$-\frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r+1)!\,x^{-(n+2r+1)}}{2^r r!\,(2n+3)(2n+5)\ldots(2n+2r+1)}$$

$$=\frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,\{(n+1)-(n+2r+1)\}\,x^{-(n+2r+1)}}{2^r r!\,(2n+3)(2n+5)\ldots(2n+2r+1)}$$

$$=\frac{2^n n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,(-2r)\,x^{-(n+2r+1)}}{2^r r!\,(2n+1)(2n+3)\ldots(2n+2r+1)}$$

$$=-\frac{2^n n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^{r-1}(r-1)!\,(2n+1)(2n+3)\ldots(2n+2r+1)}$$

(Since term corresponding to $r=0$ is zero)

$$=-\frac{2^n n!}{2n!}\sum_{s=0}^{\infty}\frac{(2+2s+2)!\,x^{-(n+2s+3)}}{2^s s!\,(2n+1)(2n+3)\ldots(2n+2s+3)}$$

Putting $r-1=s$

$$=Q'_{n+1}(x).$$

Hence $Q'_{n+1}(x)-xQ_n'(x)=(n+1)\,Q_n(x)$. **Proved.**

**(III)** $nQ'_{n+1}(x)+(n+1)\,Q'_{n-1}(x)=(2n+1)\,xQ_n'(x)$.

**Proof.** We know that

$$Q_n(x)=\frac{2^n\,n\,!}{(2n+1)\,!}\sum_{r=0}^{\infty}\frac{(n+2r)\,!\,x^{-(n+2r+1)}}{2^r\,r\,!\,(2n+3)\,(2n+5)\ldots(2n+2r+1)}$$

As in (I), we have

$$Q_n'(x)=-\frac{2^n\,n\,!}{(2n+1)\,!}\sum_{r=0}^{\infty}\frac{(n+2r+1)\,!\,x^{-(n+2r+2)}}{2^r\,r\,!\,(2n+3)\,(2n+5)\ldots(2n+2r+1)}$$

$$Q'_{n-1}(x)=-\frac{2^n\,n\,!}{(2n)\,!}\sum_{r=0}^{\infty}\frac{(n+2r)\,!\,x^{-(n+2r+1)}}{2^r\,r\,!\,(2n+1)\,(2n+3)\ldots(2n+2r-1)}$$

and $$Q'_{n+1}(x)=-\frac{2^n\,n\,!}{2n\,!}\sum_{r=0}^{\infty}\frac{(n+2r+2)\,!\,x^{-(n+2r+3)}}{2^r\,r\,!\,(2n+1)\,(2n+3)\ldots(2n+2r+3)}$$

$\therefore\quad (2n+1)\,xQ_n'(x)-(n+1)\,Q'_{n-1}(x)$

$$=-\frac{2^n\,n\,!}{2n\,!}\sum_{r=0}^{\infty}\frac{(n+2r+1)\,!\,x^{-(n+2r+1)}}{2^r\,r\,!\,(2n+3)\,(2n+5)\ldots(2n+2r+1)}$$

$$+\frac{2^n\,n\,!}{2n\,!}\sum_{r=0}^{\infty}\frac{(n+1)\,(n+2r)\,!\,x^{-(n+2r+1)}}{2^r\,r\,!\,(2n+1)\,(2n+3)\ldots(2n+2r-1)}$$

$$=-\frac{2^n\,n\,!}{2n\,!}\sum_{r=0}^{\infty}\frac{(n+2r)!\{(n+2r+1)(2n+1)-(n+1)(2n+2r+1)\}x^{-(n+2r+1)}}{2^r\,r\,!\,(2n+1)\,(2n+3)\ldots(2n+2r+1)}$$

$$=-\frac{2^n\,n\,!}{2n\,!}\sum_{r=0}^{\infty}\frac{(n+2r)\,!\,2r\,x^{-(n+2r+1)}}{2^r\,r\,!\,(2n+1)\,(2n+3)\ldots(2n+2r+1)}$$

$$=-n\,\frac{2^n\,n\,!}{2n\,!}\sum_{r=0}^{\infty}\frac{(n+2r)\,!\cdot x^{-(n+2r+1)}}{2^{r-1}\,(r-1)\,!\,(2n+1)\,(2n+3)\ldots(2n+2r+1)}$$

$$=-n\,\frac{2^n\,n\,!}{2n\,!}\sum_{r=0}^{\infty}\frac{(n+2s+2)\,!\,x^{-(n+2s+3)}}{2^s.s\,!\,(2n+1)\,(2n+3)\ldots(2n+2s+3)}$$

$=nQ'_{n+1}(x)$.

Hence $nQ'_{n+1}(x)+(n+1)\,Q'_{n-1}(x)=(2n+1)\,xQ_n'(x)$. **Proved.**

**(IV)** $xQ_n'(x)-Q'_{n-1}(x)=nQ_n(x)$.

**Proof.** We have

$$Q_n(x)=\frac{2^n\,n\,!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)\,!\,x^{-(n+2r+1)}}{2^r\,r\,!\,(2n+3)\,(2n+5)\ldots(2n+2r+1)}$$

As in (I), we have

$$Q_n'(x) = -\frac{2^n n!}{(2n+1)!} \sum_{r=0}^{\infty} \frac{(n+2r+1)!\, x^{-(n+2r+2)}}{2^r\, r!\, (2n+3)(2n+5)\ldots(2n+2r+1)}$$

and $$Q'_{n-1}(x) = -\frac{2^n n!}{2n!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}}{2^r\, r!\, (2n+1)(2n+3)\ldots(2n+2r-1)}$$

$\therefore$ $xQ_n'(x) - nQ_n(x)$

$$= -\frac{2^n n!}{(2n+1)!} \sum_{r=0}^{\infty} \frac{n\,(n+2r+1)!\, x^{-(n+2r+1)}}{2^r\, r!\, (2n+3)(2n+5)\ldots(2n+2r+1)}$$

$$-\frac{2^n n!}{(2n+1)!} \sum_{r=0}^{\infty} \frac{n\,(n+2r)!\, x^{-(n+2r+1)}}{2^r\, r!\, (2n+3)(2n+5)\ldots(2n+2r+1)}$$

$$= -\frac{2^n n!}{2n!} \sum_{r=0}^{\infty} \frac{(n+2r)!\,(n+2r+1+n)\, x^{-(n+2r+1)}}{2^n\, r!\, (2n+1)(2n+3)\ldots(2n+2r+1)}$$

$$= -\frac{2^n n!}{2n!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}}{2^n\, r!\, (2n+1)(2n+3)\ldots(2n+2r-1)}$$

$= Q'_{n-1}(x)$.

Hence $xQ_n'(x) - Q'_{n-1}(x) = nQ_n(x)$. **Proved.**

**(V)** $(x^2-1)\, Q_n'(x) = nxQ_n(x) - nQ_{n-1}(x)$.

**Proof.** Replacing $n$ by $(n-1)$ in rec. formula II, we have

$$Q_n'(x) - xQ'_{n-1}(x) = nQ'_{n-1}(x). \quad \ldots(\text{i})$$

Writing recurrence formula IV, we have

$$xQ_n'(x) - Q'_{n-1}(x) = nQ_n(x). \quad \ldots(\text{ii})$$

Multiplying (ii) by $x$ and then subtracting from (i), we have

$$(1-x^2)\, Q_n'(x) = nQ_{n-1}(x) - nxQ_n(x)$$

*i.e.* $$(x^2-1)\, Q_n'(x) = nxQ_n(x) - nQ_{n-1}(x).$$

**(VI)** $(n+1)\, Q_{n+1}(x) - (2n+1)\, xQ_n(x) + nQ_{n-1}(x) = 0$.

**Proof.** We have

$$Q_n(x) = \frac{2^n n!}{(2n+1)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}}{2^r\, r!\, (2n+3)(2n+5)\ldots(2n+2r+1)}$$

Replacing $n$ by $(n-1)$ and $(n+1)$, we have

$$Q_{n-1}(x) = \frac{2^{n-1}(n-1)!}{(2n+1)!} \sum_{r=0}^{\infty} \frac{(n+2r-1)\, x^{-(n+2r)}}{2^r\, r!\, (2n+1)(2n+3)\ldots(2n+2r-1)}$$

and $$Q_{n+1}(x) = \frac{2^{n+1}(n+1)!}{(2n+3)!} \sum_{r=0}^{\infty} \frac{(n+2r+1)!\, x^{-(n+2r+2)}}{2^r\, r!\, (2n+5)(2n+7)\ldots(2n+2r+3)}$$

$\therefore \quad -nQ_{n-1}(x)+(2n+1)\,xQ_n(x)$

$$=-\frac{2^{n-1}\,n!}{(2n-1)!}\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}}{2^r\,r!\,(2n+1)(2n+3)\ldots(2n+2r-1)}$$

$$+\frac{2^n\,n!}{2n!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r)}}{2^r\,r!\,(2n+3)(2n+5)\ldots(2n+2r+1)}$$

$$=-\frac{2^{n+1}\,n!}{(2n+3)!}\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}\,2n.(2n+2)}{2^r\,r!\,(2n+5)(2n+7)\ldots(2n+2r-1)}$$

$$+\frac{2^n\,n!}{(2n+3)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,(2n+1)\,x^{-(n+2r)}\,(n+2)}{2^r\,r!\,(2n+5)(2n+7)\ldots(2n+2r+1)}$$

$$=\frac{2^{n+1}\,(n+1)!}{(2n+3)!}\times$$

$$\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,\{-(2n+2r+1)\,n+(n+2r)(2n+1)\}\,x^{-(n+2r)}}{2^r\,r!\,(2n+5)(2n+7)\ldots(2n+2r+1)}$$

$$=\frac{2^{n+1}\,(n+1)!}{(2n+3)!}\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,(2nr+2r)\,x^{-(n+2r)}}{2^r\,r!\,(2n+5)(2n+7)\ldots(2n+2r+1)}$$

$$=(n+1)\frac{2^{n+1}\,(n+1)!}{(2n+3)!}$$

$$\times\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}}{2^{r-1}\,(r-1)!\,(2n+5)(2n+7)\ldots(2n+2r+1)}$$

$$=(n+1)\frac{2^{n+1}\,(n+1)!}{(2n+3)!}\sum_{s=0}^{\infty}\frac{(n+2s+1)!\,x^{-(n+2s+2)}}{2^s\,s!\,(2n+5)(2n+7)\ldots(2n+2s+3)}$$

Putting $r-1=s$

$=(n+1)\,Q_{n+1}(x).$

Hence $(n+1)\,Q_{n+1}(x)-(2n+1)\,xQ_n(x)+nQ_{n-1}(x)=0.$

## § 3·4. Relation between $P_n(x)$ and $Q_n(x)$.

**(I)** $(x^2-1)(Q_nP_n'-P_nQ_n')=c.$ **[Agra 82]**

**Sol.** From Legendre's equations, we have

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)\,y=0. \qquad \ldots(i)$$

Since $P_n(x)$ and $Q_n(x)$ are solutions of (i)

$$\therefore\quad (1-x^2)\frac{d^2P_n}{dx^2}-2x\frac{dP_n}{dx}+n(n+1)\,P_n=0 \qquad \ldots(ii)$$

and $(1-x^2)\dfrac{d^2Q_n}{dx^2}-2x\dfrac{dQ_n}{dx}+n(n+1)Q_n=0.$ ...(iii)

Multiplying (ii) by $Q_n$, (iii) by $P_n$ and then subtracting, we have

$$(1-x^2)\left[\frac{d^2P_n}{dx^2}Q_n-P_n.\frac{d^2Q_n}{dx^2}\right]-2x\left[\frac{dP_n}{dx}Q_n-P_n.\frac{dQ_n}{dx}\right]=0$$

or
$$\frac{\dfrac{d^2P_n}{dx^2}.Q_n-P_n.\dfrac{d^2Q_n}{dx^2}}{\dfrac{dP_n}{dx}.Q_n-P_n.\dfrac{d}{dx}Q_n}=\frac{2x}{1-x^2}$$

or
$$\frac{\dfrac{d}{dx}\left(\dfrac{dP_n}{dx}Q_n-P_n.\dfrac{d}{dx}Q_n\right)}{\dfrac{dP_n}{dx}.Q_n-P_n.\dfrac{dQ_n}{dx}}=\frac{\dfrac{d}{dx}(x^2-1)}{x^2-1}$$

Integrating we have

$$\log(Q_nP_n'-P_nQ_n')=-\log(x^2-1)+\log c$$

or
$$(x^2-1)(Q_nP_n'-P_nQ_n')=c.$$ **Proved.**

**(II)** $\dfrac{Q_n(x)}{P_n(x)}=\displaystyle\int_x^{\infty}\frac{dx}{(x^2-1)P_n^2(x)}$

**[Raj. 79, 81, 83, 86 ; Agra 81]**

**hence deduce that**

(i) $Q_0(x)=\frac{1}{2}\log\dfrac{x+1}{x-1}$ and (ii) $Q_1(x)=\dfrac{x}{2}\log\dfrac{x+1}{x-1}-1.$

[Raj. 80] **[Raj. 78, 80]**

**Sol.** In (1), we have proved that

$$Q_nP_n'-P_nQ_n'=\frac{c}{x^2-1}=\frac{c}{x^2}\left(1-\frac{1}{x^2}\right)^{-1}$$
$$=\frac{c}{x^2}\left(1+\frac{1}{x^2}+\frac{1}{x^4}+\ldots\right)$$
$$=c\left(\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+\ldots\right). \qquad \ldots(i)$$

Further we know that

$$P_n(x)=\frac{1.3\ \ (2n+1)}{n!}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}\ldots\right]$$
$$Q_n(x)=\frac{n!}{1.3\ldots(2n+1)}\left[x^{-n-1}+\frac{(n+1)(n+2)}{2.(2n+3)}x^{-n-3}\ldots\right]$$

Substituting the values of $P_n(x)$, $Q_n(x)$ and their derivatives $P_n'(x)$, $Q_n'(x)$ in (i), we have

$$Q_nP_n'-P_nQ_n'=\frac{n!}{1.3\ldots(2n+1)}\left[x^{-n-1}+\frac{(n+1)(n+2)}{2.(2n+3)}x^{-n-3}+\ldots\right]$$

$$\times \frac{1.3\ldots(2n-1)}{n!}\left[nx^{n-1}-\frac{n(n-1)(n-2)}{2\cdot(2n-1)}x^{n-3}\ldots\right]$$

$$-\frac{1\cdot3\ldots(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}\ldots\right]$$

$$\times\frac{n!}{1.3\ldots(2n+1)}\times$$

$$\left[-(n+1)x^{-n-2}-\frac{(n+1)(n+2)(n+3)}{2(2n+3)}x^{-n-4}\right]$$

$$=-c\left[\frac{1}{x^2}+\frac{1}{x^4}+\ldots\right].$$

Equating the coefficients of $\frac{1}{x^2}$ from the two sides we have

$$\frac{n!}{1\cdot3\ldots(2n+1)}\cdot\frac{1.3\ldots(2n-1)}{n!}n$$

$$+\frac{1\cdot3\ldots(2n-1)}{n!}\cdot\frac{n!\cdot(n+1)}{1\cdot3\ldots(2n+1)}=c$$

or $$c=\frac{n}{2n+1}+\frac{n+1}{2n+1}=1$$

$$\therefore\quad Q_nP'_n-P_nQ'_n=\frac{1}{x^2-1}.$$

Dividing both sides by $P_n^2$, we have

$$\frac{P_n'Q_n-Q'_nP_n}{P_n^2}=\frac{1}{(x^2-1)P_n^2}\quad\text{or}\quad\frac{d}{dx}\left(\frac{Q_n}{P_n}\right)=\frac{1}{(x^2-1)P_n^2}$$

Integrating both sides between the limits $x$ to $\infty$, we have

$$\left(-\frac{Q_n}{P_n}\right)_x^{\infty}=\int_x^{\infty}\frac{dx}{(x^2-1)P_n^2}$$

or $$-\left(\frac{Q_n}{P_n}\right)_{x\to\infty}+\frac{Q_n}{P_n}=\int_x^{\infty}\frac{dx}{(x^2-1)P_n^2}$$

or $$\frac{Q_n(x)}{P_n(x)}=\int_x^{\infty}\frac{dx}{(x^2-1)P_n^2(x)}.$$ **Proved.**

Since $$\lim_{x\to\infty}\frac{Q_n(x)}{P_n(x)}=\lim_{x\to\infty}\frac{n\text{th derivative of }Q_n(x)}{n\text{th derivative of }P_n(x)}$$

$$=\lim_{x\to\infty}\frac{\frac{n!}{1.3\ldots(2n+1)}[(-1)^n(n+1)(n+2)\ldots(n+n)x^{-(n+n+1)}+\ldots]}{\frac{1\cdot3\ldots(2n-1)}{n;}[n!+0]}$$

$=0.$

Hence $$\frac{\mathbf{Q_n(x)}}{\mathbf{P_n(x)}}=\int_x^{\infty}\frac{\mathbf{dx}}{\mathbf{(x^2-1)P_n^2(x)}}\qquad\ldots\text{(ii)}$$

**Deductions.** (i) Putting $n=0$ in (ii), we have

$$\frac{Q_0(x)}{P_0(x)}=\int_x^\infty \frac{dx}{(x^2-1)\,P_0^2(x)}$$

or $$Q_0(x)=\int_x^\infty \frac{dx}{x^2-1} \qquad \text{since } P_0(x)=1$$

or $$Q_0(x)=\tfrac{1}{2}\left(\log\frac{x-1}{x+1}\right)_x^\infty$$

or $$Q_0(x)=\tfrac{1}{2}\lim_{x\to\infty}\log\frac{x-1}{x+1}-\frac{1}{2}\log\frac{x-1}{x+1}$$

$$=\frac{1}{2}\lim_{x\to\infty}\log\frac{1-\dfrac{1}{x}}{1+\dfrac{1}{x}}-\frac{1}{2}\log\frac{x-1}{x+1}$$

$$=0-\frac{1}{2}\log\frac{x-1}{x+1}=\frac{1}{2}\log\frac{x+1}{x-1}$$ **Proved.**

(ii) Putting $n=1$ in (ii), we have

$$\frac{Q_1(x)}{P_1(x)}=\int_x^\infty \frac{dx}{(x^2-1)\,P_1^2(x)}$$

or $$Q_1(x)=x\int_x^\infty \frac{dx}{(x^2-1)\,x^2} \qquad \text{Since } P_1(x)=x$$

$$=x\int_0^\infty\left(\frac{1}{x^2-1}-\frac{1}{x^2}\right)dx=x\left[\frac{1}{2}\log\frac{x-1}{x+1}+\frac{1}{x}\right]_x^\infty$$

$$=x\left[\frac{1}{2}\left(\log\frac{x-1}{x+1}\right)_{x\to\infty}+\frac{1}{\infty}-\frac{1}{2}\log\frac{x-1}{x+1}-\frac{1}{x}\right]$$

$$=\frac{x}{2}\log\frac{x+1}{x-1}-1.$$ **Proved.**

**(III)** $\mathbf{P_nQ_{n-1}-Q_nP_{n-1}=\dfrac{1}{n}}$. **[Raj. 78, 82]**

**Solution.** From Rec. formula I for $P_n(x)$ and Rec. formula VI for $Q_n(x)$, we have

$$(2n+1)\,xP_n=(n+1)\,P_{n+1}+nP_{n-1} \qquad \text{...(i)}$$

and $$(2n+1)\,xQ_n=(n+1)\,Q_{n+1}+nQ_{n-1}. \qquad \text{...(ii)}$$

Replacing $n$ by $(n-1)$, both in (i) and (ii), we have

$$(2n-1)\,xP_{n-1}=nP_n+(n-1)\,P_{n-2} \qquad \text{...(iii)}$$

$$(2n-1)\,xQ_{n-1}=nQ_n+(n-1)\,Q_{n-2}. \qquad \text{...(iv)}$$

Multiplying (iii) by $Q_{n-1}$ and (iv) by $P_{n-1}$ and then subtracting, we have

$$n\,(P_nQ_{n-1}-Q_nP_{n-1})=(n-1)\,(P_{n-1}Q_{n-2}-Q_{n-1}P_{n-2}).$$

Hence $f_n=f_{n-1}$ where $f_n=n\,(P_nQ_{n-1}-Q_nP_{n-1})$

Similarly $f_n=f_{n-1}=f_{n-2}=\ldots=f_1=P_1Q_0-Q_1P_0$

$\therefore\ n(P_nQ_{n-1}-Q_nP_{n-1})=P_1Q_0-Q_1P_0.$ ...(v)

But $P_0=1,\ P_1=x$

$\therefore\ Q_0=P_0\int_x^\infty \frac{dx}{(x^2-1)P_0^2}$ (from II) $=\int_x^\infty \frac{dx}{(x^2-1)}$ since $P_0=1$

and $Q_1=P_1\int_x^\infty \frac{dx}{(x^2-1)P_1^2}=x\int_x^\infty \frac{dx}{(x^2-1)x^2}$

$$=x\int_x^\infty\left(\frac{1}{x^2-1}-\frac{1}{x^2}\right)dx=xQ_0-x\int_x^\infty \frac{dx}{x^2}$$

$$=xQ_0+x.\left(\frac{1}{x}\right)_x^\infty=xQ_0+x\left(0-\frac{1}{x}\right)=xQ_n-1.$$

$\therefore$ From (v), we have

$$n(P_nQ_{n-1}-Q_nP_{n-1})=xQ_0-(xQ_0-1).\ 1=1.$$

or $$P_nQ_{n-1}-Q_nP_{n-1}=\frac{1}{n}$$ **Proved.**

**(IV)** $\mathbf{P_nQ_{n-2}-Q_nP_{n-2}=\dfrac{(2n-1)}{n(n-1)}x.}$ **[Raj. 82, 85]**

**Sol.** From Recurrence formula (1) for $P_n(x)$, we have

$$(2n+1)xP_n=(n+1)P_{n+1}+nP_{n-1}. \qquad \ldots(i)$$

Also from Rec. formula VI for $Q_n(x)$, we have

$$(2n+1)xQ_n=(n+1)Q_{n+1}+nQ_{n-1}. \qquad \ldots(ii)$$

Replacing $n$ by $(n-1)$, both in (i) and (ii), we have

$$(2n-1)xP_{n-1}=nP_n+(n-1)P_{n-2} \qquad \ldots(iii)$$

and $$(2n-1)xQ_{n-1}=nQ_n+(n-1)Q_{n-2} \qquad \ldots(iv)$$

Multiplying (iv) by $P_n$ and (iii) by $Q_n$ and then subtracting, we have

$$(n-1)x(Q_{n-1}P_n-P_{n-1}Q_n)=(n-1)(Q_{n-2}P_n-Q_nP_{n-2})$$

or $$\mathbf{P_nQ_{n-2}\ \ Q_nP_{n-2}=\frac{(2n-1)x}{n(n-1)}}$$

Since $P_nQ_{n-1}-Q_nP_{n-1}=\frac{1}{n}$ from (III)

**§ 3·5. Christoffel's Second Summation Formula : To prove that**

$$\sum_{r=0}^{n}(2r+1)P_r(x)Q_r(y)=\frac{1+(n+1)[P_{n+1}(x)\ Q_n(y)-P_n(x)\ Q_{n+1}(y)]}{y-x}$$

**Proof** From Recurrence formula for $P_n$ and $Q_n$ we have

$$(2r+1)xP_r(x)=(r+1)P_{r+1}(x)+rP_{r-1}(x) \qquad \ldots(i)$$

and $$(2r+1)yQ_r(y)=(r+1)Q_{r+1}(y)+rQ_{r-1}(y) \qquad \ldots(ii)$$

Multiplying (i) by $Q_r(y)$ and (ii) by $P_r(x)$ and then subtracting we have.

$$(2r+1)(x-y)P_r(x)Q_r(y)=(r+1)[P_{r+1}(x)Q_r(y)-P_r(x)Q_{r+1}(y)]$$
$$-r[P_r(x)Q_{r-1}(y)-P_{r-1}(x)Q_r(y)]. \quad \ldots\text{(iii)}$$

Putting $r=1, 2, 3, \ldots (n-1), n$ in iii), we have

$$3(x-y)P_1(x)Q_1(y)=2[P_2(x)Q_1(y)-P_1(x)Q_2(y)]$$
$$-1.[P_1(x)Q_0(y)-P_0(x)Q_1(y)] \ldots (A_1)$$
$$5(x-y)P_2(x)Q_2(y)=3[P_3(x)Q_2(y)-P_2(x)Q_3(y)]$$
$$-2[P_2(x)Q_1(y)-P_1(x)Q_2(y)] \ldots (A_2)$$

... ... ... ... ... ... ...

... ... ... ... ... ... ...

$$(2n-1)(x-y)P_{n-1}(x)Q_{n-1}(y)=n[P_n(x)Q_{n-1}(y)-P_{n-1}(x)Q_n(y)]$$
$$-(n-1)[P_{n-1}(x)Q_{n-2}(y)-P_{n-2}(x)Q_{n-1}(y)] \ldots (A_{n-1})$$
$$(2n+1)(x-y)P_n(x)Q_n(y)=(n+1)P_{n+1}(x)Q_n(y)-P_n(x)Q_{n+1}(y)$$
$$-n[P_n(x)Q_{n-1}(y)-P_{n-1}(x)Q_n(y)] \ldots (A_n)$$

Adding $(A_1), (A_2), \ldots (A_n)$, we have

$$\sum_{r=1}^{n}(2r+1)(x-y)P_r(x)Q_r(y)=(n+1)[P_{n+1}(x)Q_n(y)-P_n(x)Q_{n+1}(y)]$$
$$-[P_1(x)Q_0(y)-P_0(x)Q_1(y)]$$
$$=(n+1)[P_{n+1}(x)Q_n(y)-P_n(x)Q_{n+1}(y)]$$
$$-[xQ_0(y)-1.\{yQ_0(y)-1\}]$$

since $Q_1(y)=y$, $Q_0(y)=1$, $P_1(x)=x$; $P_0(x)=1$

$$=-1+(n+1)[P_{n+1}(x)Q_n(y)-P_n(x)Q_{n+1}(y)]$$
$$-(x-y)P_0(x)P_0(y) \text{ since } P_0(x)=1$$

or $$\sum_{r=0}^{n}(2r+1)(y-x)P_r(x)Q_r(y)=1-(n+1)[P_{n+1}(x)Q_n(y)$$
$$-P_n(x)Q_{n+1}(y)]$$

or $$\sum_{r=0}^{n}(2r+1)P_r(x)Q_r(y)$$

$$=\frac{1-(n+1)[P_{n+1}(x)Q_n(y)-P_n(x)Q_{n+1}(y)]}{x-y}$$

**Proved.**

**§ 3·6. Assuming $P_n$ as a solution of Legendre's equation, show that the complete solution of this Legendre's equation is given by**

**$AP_n+BQ_n$ where $Q_n=cP_n\int\frac{dx}{(1-x^2)P_n^2}$ c being a const.**

**Proof.** From Legendre's equation, we have

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0. \quad \ldots\text{(i)}$$

Let $y=uP_n$ be the complete solution of the equation (i)

where $u$ is a function of $x$,

so that $\dfrac{dy}{dx}=u\dfrac{dP_n}{dx}+P_n\dfrac{du}{dx}$

and $\dfrac{d^2y}{dx^2}=u\dfrac{d^2P_n}{dx^2}+2\dfrac{du}{dx}\dfrac{dP_n}{dx}+P_n.\dfrac{d^2u}{dx^2}$

Substituting these values in (i), we have

$$(1-x^2)\left[u\frac{d^2P_n}{dx^2}+2\frac{du}{dx}.\frac{dP_n}{dx}+P_n.\frac{d^2u}{dx^2}\right]-2x\left[u.\frac{dP_n}{dx}+P_n.\frac{du}{dx}\right]$$
$$+n(n+1)\,uP_n=0$$

or $$(1-x^2)\left(P_n\frac{d^2u}{dx^2}+2\frac{du}{dx}.\frac{dP_n}{dx}\right)+u.\left((1-x^2)\frac{d^2P_n}{dx^2}-2x\frac{dP_n}{dx}\right.$$
$$\left.+n(n+1)P_n\right)-2xP_n\frac{du}{dx}=0$$

or $$(1-x^2)\left(P_n\frac{d^2u}{dx^2}+2\frac{du}{dx}.\frac{dP_n}{dx}\right)-2xP_n\frac{du}{dx}=0.$$

Since $P_n$ is the solution of (i)

or $$P_n\frac{d^2u}{dx^2}+2\frac{du}{dx}\frac{dP_n}{dx}=\frac{2x}{1-x^2}.P_n\frac{du}{dx}$$

or $$\frac{\dfrac{d^2u}{dx^2}}{\dfrac{du}{dx}}+2\frac{\dfrac{dP_n}{dx}}{P_n}=\frac{2x}{1-x^2}$$

Integrating, we have

$$\log\frac{du}{dx}+2\log P_n=-\log(1-x^2)+\log D$$

where $D$ is a constant.

or $$\frac{du}{dx}P_n^2(1-x^2)=D \quad\text{or}\quad \frac{du}{dx}=\frac{D}{(1-x^2)P_n^2}.$$

Integrating, $u=D\displaystyle\int\frac{dx}{(1-x^2)P_n^2}+A$

where $A$ is a constant.

Hence the complete solution of (i) is given by

$$y=\left[D\int\frac{dx}{(1-x^2)P_n^2}+A\right]P_n$$
$$=AP_n+\frac{D}{c}cP_n\int\frac{dx}{(1-x^2)P_n^2}=AP_n+BQ_n$$

where $B=\dfrac{D}{c}$ (const) and $Q_n=cP_n\displaystyle\int\frac{dx}{(1-x^2)P_n^2}$. **Proved.**

**Examples.**

**Ex. 1.** *Prove that*

$$(2n+1)(1-x^2)\,Q'_n(x)=n(n+1)\,[Q_{n-1}(x)-Q_{n+1}(x)].$$

**Sol.** $Q_n(x)$ is the solution of the Legendre's equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+n(n+1)y=0. \qquad \text{...(i)}$$

$$\therefore \quad \frac{d}{dx}[(1-x^2)Q'_n]=-n(n+1)Q_n. \qquad \text{...(ii)}$$

Integrating both sides between the limits $\infty$ to $x$, we have

$$\left[(1-x^2)\cdot Q'_n\right]_\infty^x=-n(n+1)\int_\infty^x Q_n\,dx$$

or $$(1-x^2)Q'_n-(Q'_n-x^2Q'_n)_{x=\infty}=-n(n+1)\int_\infty^x Q_n\,dx$$

or $$(1-x^2)Q'_n=-n(n+1)\int_\infty^x Q_n\,dx \qquad \text{...(iii)}$$

Since $(Q'_n)_{x=\infty}=0$ and $(x^2Q'_n)_{x=\infty}=0$.

Again by recurrence formula (1) for $Q_n(x)$, we have

$$Q'_{n+1}-Q'_{n-1}=(2n+1)Q_n. \qquad \text{...(iv)}$$

Integrating both sides between the limits $\infty$ to $x$, we have

$$\left(Q_{n+1}-Q_{n-1}\right)_\infty^x=(2n+1)\int_\infty^x Q_n\,dx$$

or $$Q_{n+1}-Q_{n-1}=(2n+1)\int_\infty^x Q_n\,dx \qquad \text{...(v)}$$

Since $(Q_{n+1})_{x=\infty}=0$ and $(Q_{n-1})_{x=\infty}=0$.

Eliminating $\int_\infty^x Q_n\,dx$ from (iii) and (v) ; we have

$$(1-x^2)Q'_n=-\frac{n(n+1)}{(2n+1)}(Q_{n+1}-Q_{n-1})$$

or $(2n+1)(1-x^2)Q'_n(x)=n(n+1)[(Q_{n-1}(x)-Q_{n+1}(x)]$. **Proved.**

**Ex. 2.** *Prove that*

$$Q_n(x)=2^n n!\int_x^\infty dx\int_x^\infty dx\,..\int_x^\infty (x^2-1)^{-n-1}\,dx.$$

**Sol.** $$(x^2-1)^{-n-1}=x^{-2n-2}\left(1-\frac{1}{x^2}\right)^{-n-1}$$

$$=x^{-2n-2}\left(1+(n+1)\cdot\frac{1}{x^2}+\frac{(n+1)(n+2)}{2!}\cdot\frac{1}{x^4}+\ldots\right)$$

$$=x^{-(2n+2)}+(n+1)x^{-(2n+4)}+\frac{(n+1)(n+2)}{2!}x^{-(2n+6)}+\ldots$$

Integrating both sides w.r.t. '$x$', between the limits $x$ to $\infty$, we nave

$$\int_x^\infty (x^2-1)^{-n-1}\,dx=\left[-\frac{x^{-(2n+1)}}{(2n+1)}-\frac{(n+1)}{2n+3}x^{-(2n+3)}\right.$$
$$\left.-\frac{(n+1)(n+2)}{2!(2n+5)}\cdot x^{-(2n+5)}\ldots\right]_x^\infty$$

$$=\frac{x^{-(n+1)}}{(2n+1)}+\frac{(n+1)}{(2n+3)}x^{-(2n+3)}+\frac{(n+1)(n+2)}{2!(2n+5)}x^{-(2n+5)} \quad ...(i)$$

Integrating (i) $n$ times, w.r.t. '$x$' between the same limits $x$ to $\infty$, we have

$$\int_x^\infty dx\int_x^\infty dx...\int_x^\infty (x^2-1)^{-n-1}\,dx=\frac{x^{-(n+1)}}{(2n+1)(2n)...(n+1)}$$

$$+\frac{(n+1)\,x^{-(n+3)}}{(2n+3)(2n+2)...(n+3)}$$

$$+\frac{(n+1)(n+2)\,x^{-(n+5)}}{2!(2n+5)(2n+3)...(n+5)}+.....$$

$$=\frac{n!\,x^{-n-1}}{\{1.3...(2n+1)\}\{2.4...2n\}}$$

$$+\frac{n!(n+1)(n+2)(n+1)\,x^{-n-3}}{\{1.3...(2n+1)(2n+3)\}\{2.4...2n(2n+2)\}}$$

$$+\frac{n!(n+1)(n+2)\,x^{-n-5}(n+4)(n+3)(n+2)(n+1)}{2!\{1.3...(2n+1)(2n+3)(2n+5)\}\{2.4...(2n)(2n+2)(2n+4)\}}+...$$

$$\therefore\quad 2^n n!\int_x^\infty dx\int_x^\infty dx...\int_x^\infty (x^2-1)^{-n-1}\,dx=\frac{n!}{1.3...(2n+1)}$$

$$\left[x^{-n-1}+\frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3}\right.$$

$$\left.+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5}+...\right]=Q_n(x).$$

*i.e.* $$Q_n(x)=2^n n!\int_x^\infty dx.\int_x^\infty dx...\int_x^\infty (x^2-1)^{-n-1}\,dx.$$ **Pd.**

**Ex. 3.** *Prove that*

$$\frac{d^{n+1}Q_n}{dx^{n+1}}=-\frac{(-2)^n n!}{(x^2-1)^{n+1}}.$$ **[Agra 77]**

**Sol.** We know that

$$Q_n(x)=\frac{n!}{1.3.5...(2n+1)}\left[x^{-n-1}+\frac{(n+1)(n+2)}{2.(2n+3)}x^{-n-2}\right.$$

$$\left.+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5}+...\right]$$

Differentiating both sides $(n+1)$ times, w.r.t. '$x$'. we have

$$\frac{d^{n+1}Q_n}{dx^{n+1}}=\frac{n!}{1.3...(2n+1)}\left[(-1)^{n+1}(n+1)(n+2)...(2n+1)\,x^{-2n-2}\right.$$

$$+(-1)^{n+1}\frac{(n+1)(n+2)(n+3)...(2n+3)\,x^{-2n-4}}{2.(2n+3)}$$

$$\left.+(-1)^{n+1}\frac{(n+1)(n+2)...(2n+5)}{2.4(2n+3)(2n+5)}x^{-2n-6}+...\right]$$

$$=\frac{(-1)^{n+1}(2n+1)!\,x^{-2n-2}}{1.3\ldots(2n+1)}\left[1+\frac{(2n+2)}{3}.x^{-2}\right.$$

$$\left.+\frac{(2n+2)(2n+3)(2n+4)(2n+5)}{2.4.(2n+3)(2n+5)}x^{-4}+\ldots\right]$$

$$=(-1)^{n+1}\,2.4.6\ldots(2n)\,x^{-2n-2}\left[1+(n+1)\frac{1}{x^2}\right.$$

$$\left.+\frac{(n+1)(n+2)}{2!}\frac{1}{x^4}+\ldots\ldots\right]$$

$$=(-1)^{n+1}\,2^n\,n!\,x^{-2n-2}\left(1-\frac{1}{x^2}\right)^{-n-1}$$

$$=(-1)(-2)^n\,n!\,(x^2-1)^{-n-1}=-\frac{(-2)^n.n!}{(x^2-1)^{n+1}}$$ **Pd.**

**Ex. 4.** *Show that*

$$Q_2(x)=\tfrac{1}{2}P_2(x)\log\frac{x+1}{x-1}-3x/2.$$

**Sol.** Putting $n=1$ in Rec. formula VI for $Q_n(x)$, we have

$$2Q_2=3xQ_1-Q_0$$

$$=3x\left[\frac{x}{2}\log\frac{x+1}{x-1}-1\right]-\tfrac{1}{2}\log\frac{x+1}{x-1}$$

(Substituting the values of $Q_0$ and $Q_1$ from § (3.4 II))

$$=\frac{(3x^2-1)}{2}\log\frac{x+1}{x-1}-3x$$

$$=P_2(x)\log\frac{x+1}{x-1}-3x \quad \left[\text{Since } P_2(x)=\tfrac{1}{2}(3x^2-1)\right]$$

$$\therefore\quad Q_2(x)=\tfrac{1}{2}P_2(x)\log\frac{x+1}{x-1}-3x/2.$$ **Proved.**

# 4
# Hypergeometric Functions

## § 4·1. Hypergeometric function. [Meerut 74 (S), 75, 76]

The Poebhammer symbol $(\alpha)_r$ is defind by

$$(\alpha)_r = \alpha\,(\alpha+1)\,(\alpha+2)\ldots\ldots(\alpha+r-1)$$

$$= \frac{\Gamma\,(\alpha+r)}{\Gamma\,(\alpha)} \qquad \text{(where } r \text{ is a + ve integer)}$$

and $(\alpha)_0 = 1$.

The general hypergeometric function.

$${}_mF_n\,(\alpha_1, \alpha_2\ldots\ldots\alpha_m\,;\,\beta_1, \beta_2\,\ldots\ldots,\beta_n\,;\,x)$$

is defined by

$${}_mF_n\,(\alpha_1, \alpha_2, \ldots\ldots\alpha_m\,;\,\beta_1, \beta_2\ldots\ldots\beta_n\,;\,x)$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha_1)_r\,(\alpha_2)_r\ldots(\alpha_m)_r}{(\beta_1)_r\,(\beta_2)_r\ldots(\beta_n)_r}\;\frac{x^r}{r\,!}$$

Another notation often used for general hypergeometric function is

$${}_mF_n = \left[\begin{matrix}\alpha_1, \alpha_2\ldots\alpha_m\,; \\ \beta_1, \beta_2,\ldots\beta_n\end{matrix}\;\; x\right]$$

Here we shall consider only two cases.

**Case I.** $m=n=1$.

In this case the hypergeometric function.

${}_1F_1\,(\alpha, \beta, x)$ will be called the **confluent hypergeometric function or kummer function.**

Thus

$${}_1F_1\,(\alpha, \beta, x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\beta)_r} \cdot \frac{x^r}{r\,!}.$$

The confluent hypergeometric function ${}_1F_1\,(\alpha, \beta, x)$ is also denoted by $M\,(\alpha, \beta, x)$.

**Case II.** $m=2, n=1$.

In this case the hypergeometric function ${}_2F_1\,(\alpha_1, \alpha_2\,\beta, x)$ will merely be called the **bypergeometric function.**

Thus

$$_2F_1(\alpha_1, \alpha_2, \beta, x) = \sum_{r=0}^{\infty} \frac{(\alpha_1)_r\,(\alpha_2)_r}{(\beta)_r} \cdot \frac{x^r}{r!}.$$

Thus the sum of the **Hypergeometric series** (given in § 4·3)is called the **hypergeometric function.**

The hypergeometric function $_2F_1(\alpha_1, \alpha_2, \beta, x)$ is also denoted by $F(\alpha_1, \alpha_2, \beta, x)$.

§ 4·2. **Gauss's Hypergeometric equation.** **[Raj. 80, 81, 82]**

The differential equation

$$x(1-x)\frac{d^2y}{dx^2} + [c-(a+b+1)\,x]\frac{dy}{dx} - aby = 0$$

is known as **Gauss's hypergeometric equation** or simple hypergeometric equation or Gauss's equation.

§ 4·3. **The Hypergeometric Series.** The series

$$1 + \frac{ab}{1!\,c}x + \frac{a(a+1)\,b(b+1)}{2!\,c(c+1)}x^2 + \frac{a(a+1)(a+2)\,b(b+1)(b+2)}{3!\,c(c+1)(c+2)}x^3 + \ldots$$

is called the **hypergeometric series** and is denoted by $_2F_1(a, b, c, x)$

It can be shown that the series converges absolutely if $|x| < 1$ and if $|x| = 1$, and series converges absolutely if $c-a-b > 0$ or $c > a+b$.

Hypergeometric series is frequently used in connection with the theory of Spherical Harmonics. Therefore some of its properties are given bere.

§ 4·4. **Particular Cases of Hypergeometric Series** Here we are giving some well known series denoted by hypergeometric function.

1. $$(1+x)^n = 1 + nx + \frac{n(n-1)}{3!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$= 1 + \frac{(-n).1}{1!\,1}(-x) + \frac{(-n)(-n+1)\,(1)(1+1)}{2!.1\,(1+1)}(-x)^2 + \frac{(-n)(-n+1)(-n+2)\,1\,(1+1)(1+2)}{3!.\,1.\,(1+1).\,(1+2)}(-x)^2 \ldots$$

$$= {}_2F_1(-n, 1, 1, -x).$$ [Meerut 74 (S)]

2. $\log(1+x)=x-x^2/2+x^3/3-x^4/4\ldots$

$$=x[1-x/2+x^2/3-x^3/4\ldots]$$

$$=x\left[1+\frac{1.1}{1\,!.2}(-x)+\frac{1\,(1+1).1.(1+1)}{2\,!\,2\,(2+1)}(-x)^2+\frac{1.(1+1).(1+2).1.(1+1).(1+2)}{3\,!\,2\,(2+1)\,(2+2)}(-x)^3+\ldots\right]$$

$$=x\;{}_2F_1(1,1,2,-x).$$ [Meerut 81 (P); Kanpur 83]

3. $\tan^{-1}x=x-x^3/3+x^5/5-x^7/7+\ldots$

$$=x[1-x^2/3+x^4/5-x^6/7+\ldots]$$

$$=x\left[1+\frac{1/2.1}{1\,!.3/2}(-x^2)+\frac{1/2\,(1/2+1).1\,(1+1)}{2\,!\,3/2\,(3/2+1)}(-x^2)^2+\ldots\right]$$

$$=x.{}_2F_1(1/2,1,3/2,-x^2)$$ [Meerut 81]

4. $\sin^{-1}x=x+1^2\dfrac{x^3}{3\,!}+1^2.3^2\dfrac{x^5}{5\,!}+1^23^2.5^2\dfrac{x^7}{7\,!}+$

$$=x\left[1+1^2.\frac{x^2}{3\,!}+1^2.3^2\frac{x^4}{4\,!}+1^2.3^2.5.\frac{x^6}{7\,!}+\ldots\right]$$

$$=x\left[1+\frac{1/2.1/2}{1\,!\,3/2}x^2+\frac{1/2.3/2.3/2}{2\,!\,3/2\,(3/2+1)}(x^2)^2+\ldots\right]$$

$$=x.{}_2F_1(1/2,1/2,3/2,x^2)$$

[Meerut 74(S), 79 (S) ; Kanpur 83]

## § 4·5. Different forms of Hypergeometric Function.

The hypergemetric function $F(a, b, c, x)$ can be put in the following forms as well :

$$F(a,b,c,x)=(1-x)^{c-a-b}\,{}_2F_1(c-a,c-b,c-x) \quad \ldots\text{(i)}$$

$$=(1-x)^{-a}\,{}_2F_1\left(a,c-b,c,\frac{x}{x-1}\right) \quad \ldots\text{(ii)}$$

$$=(1-x)^{-b}\,{}_2F_1\left(b,c-a,c,\frac{x}{x-1}\right) \quad \ldots\text{(iii)}$$

## § 4·6. Solution of the Hypergeometric Equation.

[Raj. 80, 81, 82; Meerut 84, 88; Agra 84]

Dividing the hypergeometric differential equation throughout by $(x^2-x)$, we have

$$\frac{d^2y}{dx^2}+X_1\frac{dy}{dx}+X_2y=0$$

where $X_1=\dfrac{(1+\alpha+\beta)\,x-\gamma}{(x^2-x)}$ and $X_2=\dfrac{\alpha\beta}{(x^2-x)}$

Hence $X_1\to\infty$ when $x=0$ or 1, or $\infty$

and $X_2\to\infty$ when $x=0$ or 1, or $\infty$

Hence $x=0$, $x=1$ and $x=\infty$ are the singular points.

Since $(x^2-x)\,X_1$ and $(x^2-x)^2\,X_2$ remain finite for all finite values of $x$ therefore hypergeometric differential equation can be solved by series integration method.

The hypergeometric differential equation is

$$x\,(1-x)\frac{d^2y}{dx^2}+[\gamma-(\alpha+\beta+1).x]\frac{dy}{dx}-\alpha\beta y=0 \qquad \text{...(i)}$$

We shall develop the series about $x=0$, $x=1$ and $x=\infty$.

**(a) When x=0.** Let us consider the solution of (i) in ascending powers of $x$ as

$$y=\sum_{r=0}^{\infty} a_r\, x^{k+r}$$

so that
$$\frac{dy}{dx}=\sum_{r=0}^{\infty} a_r\,(k+r)\,x^{k+r-1}$$

and
$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r\,(k+r)\,(k+r-1)\,x^{k+r-2} \qquad \text{...(ii)}$$

Substituting in (i), we have

$$\sum_{r=0}^{\infty} a_r\,[(x^2-x)\,(k+r)\,(k+r-1)\,x^{k+r-2}$$
$$+\{(1+\alpha+\beta)\,x-\gamma\}\,(k+r)\,x^{k+r-1}+\alpha\beta x^{k+r}]=0$$

and
$$\sum_{r=0}^{\infty} a_r\,[\{(k+r)^2+(\alpha+\beta)\,(k+r)+\alpha\beta\}\,x^{k+r}$$
$$-(k+r)\,(k+r+\gamma-1)\,x^{k+r-1}]=0. \qquad \text{...(ii)}$$

Now (iii) being an identity, we can equate to zero the coefficients of various powers of $x$.

$\therefore$ Equating to zero the coefficient of lowest power of $x$ *i.e.* of $x^{k-1}$, we have

$$a_0 k\,(k+\gamma-1)=0.$$

Now $a_0\neq 0$, as it is the coefficient of the first term with which we start to write the series.

$$\therefore\quad k=0$$

or
$$k=1-\gamma. \qquad \text{...(iv)}$$

Equating to zero the coefficient of the next higher power of $x$, *i.e.* of $x^k$, we have

$$a_0\,\{k^2+(\alpha+\beta)\,k+\alpha\beta\}-(k+1)\,(k+\gamma)\,a_1=0.$$

$$\therefore \quad a_1 = \frac{k^2+(\alpha+\beta)\,k+\alpha\beta}{(k+1)\,(k+\gamma)}\,a_0$$

or $$a_1 = \frac{(k+\alpha)\,(k+\beta)}{(k+1)\,(k+\gamma)}\,a_0.$$

Again equating to zero the coefficient of the general term *i.e.* of $x^{k+r}$, we have

$$a_r\,\{(k+r)^2+(\alpha+\beta)\,(k+r)+\alpha\beta\}$$
$$-(k+r+1)\,(k+r+\gamma)\,a_{r+1}=0$$

$$\therefore \quad a_{r+1} = \frac{(k+r+\alpha)\,(k+r+\beta)}{(k+r+1)\,(k+r+\gamma)}\,a_r \qquad \text{...(v)}$$

**Case I.** When $k=0$.

From (v), we have

$$a_{r+1} = \frac{(\alpha+r)\,(\beta+r)}{(r+1)\,(\gamma+r)}\,a_r.$$

$\therefore$ Putting $r=0, 1, 2, \ldots$

$$a_1 = \frac{\alpha\cdot\beta}{1\cdot\gamma}$$

$$a_2 = \frac{(\alpha+1)\,(\beta+1)}{2.(\gamma+1)}\,a_1$$

$$= \frac{\alpha.(\alpha+1)\cdot\beta\,(\beta+1)}{2\,!\,\gamma\,(\gamma+1)}\,a_0.$$

$$a_3 = \frac{(\alpha+2)\,(\beta+2)}{3\,!\,(\gamma+2)}\,a_2.$$

$$= \frac{\alpha\,(\alpha+1)\,(\alpha+2)\,\beta\,(\beta+1)\,(\beta+2)}{2\,!\cdot\gamma\,(\gamma+1)\,(\gamma+2)}\,a_0$$

etc.

$\therefore$ From (ii), we have

$$y=a_0x^k+a_1x^{k+1}+a_2x^{k+2}+\ldots$$

or $$y=a_0\left[1+\frac{\alpha\,\beta}{1\,\gamma}\,x+\frac{\alpha\,(\alpha+1)\,\beta\,(\beta+1)}{2\,!\,\gamma\,(\gamma+1)}\,x^2\right.$$

$$\left.+\frac{\alpha\,(\alpha+1)\,(\alpha+2)\,\beta\,(\beta+1)\,(\beta+2)}{3\,!\,\gamma\,(\gamma+1)\,(\gamma+2)}\,x^2+\ldots+\frac{(\alpha)_n\,(\beta)_n}{n\,!\,(\gamma)_n}\,x^n+\ldots\right]$$

Since $k=0$

$$=a_0\sum_{n=0}^{\infty}\frac{(\alpha)_n\,(\beta)_n}{n\,!\,(\gamma)_n}\,x^n \qquad \text{where } (\alpha)_n=\alpha\,(\alpha+1)\ldots(\alpha+n-1)$$

$$=a_0\cdot{}_2F_1\,(\alpha,\,\beta,\,\gamma,\,x).$$

If $a_0=1$, this series called the **hypergeometric series** and its sum $y$ denoted by ${}_2F_1\,(\alpha, \beta, \gamma, x)$ is called the **hypergeometric function.** [Meerut 84 (P); Kanpur 83]

In case $\alpha=1$ and $\beta=\gamma$ then the hypergeometric series becomes

$$y=1+x+x^2+x^3+\ldots$$

which is a geometric series. That is why it is called **hypergeometric series.**

**Case II.** When $k=1-\gamma$ from (v) we have

$$a_{r+1}=\frac{(1-\gamma+r+\alpha)\,(1-\gamma+r+\beta)}{(1-\gamma+r+1)\,(1-\gamma+r+\gamma)}\cdot a_r$$

or
$$a_{r+1}=\frac{(\alpha'+r)\,(\beta'+r)}{(\gamma'+r)\,(r+1)}\,a_r$$

where
$$\alpha'=1-\gamma+\alpha$$
$$\beta'=1-\gamma+\beta$$

and
$$\gamma'=2-\gamma$$

Putting $r=0, 1, 2, \ldots$ we have

$$a_1=\frac{\alpha'\cdot\beta'}{1\cdot\gamma'}\,a^0$$

$$a_2=\frac{(\alpha'+1)\,(\beta'+1)}{2\cdot(\gamma'+1)}\,a_1$$

$$=\frac{\alpha'\,(\alpha'+1)\,\beta'\,(\beta'+1)}{2\,!\,\gamma'\,(\gamma'+1)}\,a_0$$

$$a_3=\frac{(\alpha'+2)\,(\beta'+2)}{3\,(\gamma'+2)}\,a_2$$

$$=\frac{\alpha'\,(\alpha'+1)\,(\alpha'+2)\,\beta'\,(\beta'+1)\,(\beta'+2)}{3\,!\,\gamma'\,(\gamma'+1)\,(\gamma'+2)}\,a_0$$

etc.

$\therefore$ from (iii), we have

$$y=a_0x^k+a_1x^{k+1}+a_2x^{k+2}+\ldots$$

$$=a_0x^k\left[1+\frac{\alpha'.\beta'}{1\cdot\gamma'}\,x+\frac{\alpha'\,(\alpha'+1)\,\beta'\,(\beta'+1)}{2\,!\,\gamma'(\gamma'+2)}x^2+\ldots\right]$$

$$=b_0x^{1-\gamma}\left[1-\frac{\alpha'\cdot\beta'}{1\cdot\gamma'}\,x+\frac{\alpha'(\alpha'+1)\,\beta'(\beta'+1)}{2\,!\,\gamma'(\gamma'+1)}\,x^2+\ldots+\ldots\frac{(\alpha')_n\,(\beta')_n}{n\,!\,(\gamma')_n}\,x^n+\ldots\right]$$

$$=a_0x^{1-\gamma}\sum_{n=0}^{\infty}\frac{(\alpha')_n(\beta')_n}{n\,!\,(\gamma')_n}$$

$$=a_0x^{1-\gamma}\,{}_2F_1\,(\alpha',\beta'\,;\,\gamma'\,;\,x)$$

$$=x^{1-\gamma}\,{}_2F_1\,(\alpha-\gamma+1,\beta-\gamma+1\,;\,2-\gamma\,;\,x) \qquad \ldots\text{(vii)}$$

as $a_0=1$.

Thus we get two independent particular solutions (vi) and (vii) of the hypergeometric differential equation about $x=0$.

Hence the general solution is given by

$$y=A\ {}_2F_1(\alpha, \beta ; \gamma ; x)$$
$$+Bx^{1-\gamma}\ {}_2F_1(\alpha-\gamma+1, \beta-\gamma+1 ; 2-\gamma ; x) \quad \ldots\text{(viii)}$$

[Meerut 84, 88 ; Agra 84]

**Note.** $\gamma$ should not be a negative integer or unity. Since when $\gamma=1$, the solution (vii) is identical with solution (vi) and when $\gamma=-$ve integer $a_n=\infty$ so ${}_2F_1(\alpha, \beta; \gamma; x)$ can not be obtained in this case.

**(b) When x=1.** If $1-x=z$.

Then $$\frac{dz}{dx}=-1.$$

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=-\frac{dy}{dz}$$

and $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{dy}{dz}\right)=-\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$
$$=\frac{d^2y}{dz^2}.$$

Thus equation (i) reduces to

$$z(1-z)\frac{d^2y}{dz^2}+[\gamma-(\alpha+\beta+1)(-z+1)]\left(-\frac{dy}{dz}\right)-\alpha\beta y=0$$

or $$z(1-z)\frac{d^2y}{dz^2}+[(\alpha+\beta-\gamma+1)-(\alpha+\beta+1)z]\frac{dy}{dz}-\alpha\beta y=0,$$

which is identical with (i) with $\gamma$ replaced by $\alpha+\beta-\gamma+1$ and $x$ by $z=1-x$.

Hence from (viii), the required solution is

$$y=A\ {}_2F_1(\alpha, \beta ; \alpha+\beta-\gamma+1; 1-x)$$
$$+B(1-x)^{\gamma-\alpha-\beta}\ {}_2F_1(\gamma-\beta, \gamma-\alpha ; \gamma-\alpha-\beta+1; 1-x)$$

or $$y=A\ {}_2F_1(\alpha, \beta; \alpha+\beta-\gamma+1, 1-x)$$
$$+B(1-x)^{\gamma-\beta-\alpha}\ {}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma-\alpha-\beta+1; 1-x) \quad \ldots\text{(ix)}$$

**(c) When x=∞.** In this case we consider the solution of (i) as

$$y=\sum_{r=0}^{\infty} a_r x^{-k-r}$$

Substituting in (i), we have

$$\sum^{\infty} a_r [(-k-r)(-k-r-1+\gamma)x^{-k-r-1}$$
$$-(k-r+\alpha)(-k-r+\beta)x^{-k-r}]=0. \quad \ldots\text{(x)}$$

Equating to zero coefficient of $x^{-k}$, we have

$$a_0(-k+\alpha)(-k+\beta)=0$$

$$\because \quad a_0\neq 0.$$

$$\therefore \quad \left.\begin{array}{l} k=\alpha \\ \text{or} \quad k=\beta \end{array}\right\}. \qquad \ldots(\text{xi})$$

Again equating to zero the coefficient of the general term *i.e.*, of $x^{-k-r}$, we have

$$a_{r-1}(-k-r+1)(-k-r+\gamma)-a_r(-k-r+\alpha)(-k-r+\beta)=0$$

$$\therefore \quad a_r=\frac{(-k-r+1)(-k-r+\gamma)}{(-k-r+\alpha)(-k-r+\beta)}a_{r-1}$$

**Case I.** When $k=\alpha$,

$$a_r=\frac{(r+\alpha-1)(r+\alpha-\gamma)}{r(r+\alpha-\beta)}a_{r-1}.$$

Putting $r=1, 2, 3,$

$$a_1=\frac{\alpha.(1+\alpha-\gamma)}{1.(1+\alpha-\beta)}a_0$$

$$a_2=\frac{(\alpha+1)(2+\alpha-\gamma)}{2.(2+\alpha-\beta)}a_1$$

$$=\frac{(\alpha+1)(2+\alpha-\gamma)}{2.(2+\alpha-\beta)}\cdot\frac{\alpha.(1+\alpha-\gamma)}{1.(1+\alpha-\beta)}a_0$$

$$=\frac{\alpha(\alpha+1).(1+\alpha-\gamma)(2+\alpha-\gamma)}{2!(1+\alpha-\beta)(2+\alpha-\beta)}a_0$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$a_n=\frac{[\alpha(\alpha+1)\ldots(\alpha+r-1).(1+\alpha-\gamma)(1+1+\alpha-\gamma)\ldots\ldots(1+r-1+\alpha-\gamma)\ldots]}{2!(1+\alpha-\beta)(1+1+\alpha-\beta)\ldots(1+r-1+\alpha-\beta)}a_0$$

$$=\frac{(\alpha)_r(1+\alpha-\gamma)_r}{r!(\alpha-\beta+1)_r}a_0$$

$$\therefore \quad y=\sum_{r=0}^{\infty}\frac{(\alpha)_r(1+\alpha-\gamma)_r}{r!(\alpha-\beta+1)_r}x^{-\alpha-r} \text{ when } a_0=1$$

or

$$y=x^{-\alpha}\sum_{r=0}^{\infty}\frac{(\alpha)_r(\alpha-\gamma+1)_r}{r!(\alpha-\beta+1)_r}\left(\frac{1}{x}\right)^r$$

$$y=x^{-\alpha}\,{}_2F_1\left(\alpha, \alpha-\gamma+1; \alpha-\beta+1; \frac{1}{x}\right)$$

**Case II.** When $k=\beta$

$$y=x^{-\beta}\,{}_2F_1\left(\beta, \beta-\gamma+1; \beta-\alpha+1; \frac{1}{x}\right).$$

Hence the general solution is

$$y = Ax^{-\alpha}\,{}_2F_1\left(\alpha,\ \alpha-\gamma+1\ ;\ \alpha-\beta+1,\ \frac{1}{x}\right)$$
$$+B\,x^{-\beta}\,{}_2F_1\left(\beta,\ \beta-\gamma+1\ ;\ \beta+\alpha+1,\frac{1}{x}\right) \quad \text{...(xii)}$$

**§ 4·7. Linear relation between the solution of the hypergeometric eqnation.**

The series in the solution (vii) (§ 4·6) are convergent if $|x| < 1$ *i.e.* in the interval $(-1, 1)$ and the series in the solution (ix), are convergent in the interval (0, 2). Thus in the interval (0, 1) all the four series in (viii) and (ix) are convergent. Since only two solutions of the differential equation (i) are linearly independent there must exist a linear relation between the solutions (viii) and (ix) in the interval (0, 1).

So let

$${}_2F_1\,(\alpha,\beta;\gamma;x) = A\,{}_2F_1\,(\alpha,\beta;\alpha-\beta-\gamma+1;1-x),$$
$$+B\,(1-x)^{\gamma-\alpha-\beta}\,{}_2F_1\,(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-x) \quad \text{...(1)}$$

To determine the constants $A$ and $B$; we suppose that $\alpha+\beta<\gamma<1$, so that three series

$${}_2F_1\,(\alpha,\beta;\gamma;1),\ {}_2F_1\,(\alpha,\beta;\alpha+\beta-\gamma+1;1),$$
$${}_2F_1\,(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1)$$

are convergent.

Then putting $x=0$ and $x=1$ in (1), we have

$$1 = A\,{}_2F_1\,(\alpha,\beta;\alpha+\beta-\gamma+1;1)$$
$$+B\,{}_2F_1\,(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1)$$

and
$${}_2F_1\,(\alpha,\beta;\gamma;1) = A$$

$$A = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}$$

and
$$1 = A\,\frac{\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)\,\Gamma(\alpha-\gamma+1)}$$
$$+B\,\frac{\Gamma(\gamma-\alpha-\beta+1)\,\Gamma(1-\gamma)}{\Gamma(1-\beta)\,\Gamma(1-\alpha)}$$

(from Gauss Theorem § 4·11)

or
$$1 = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\cdot\frac{\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)\,\Gamma(\alpha-\gamma+1)}$$
$$+B\,\frac{\Gamma(\gamma-\alpha-\beta+1)\,\Gamma(1-\gamma)}{\Gamma(1-\beta)\,\Gamma(1-\alpha)}$$

or $$1=\frac{\{\Gamma(\gamma)\,\Gamma(1-\gamma)\}\,\{\Gamma(\gamma-\alpha-\beta)\,\Gamma(\alpha+\beta-\gamma+1)}{\{\Gamma(\gamma-\alpha).\Gamma(\alpha-\gamma+1)\}.\{\Gamma(\gamma-\beta)\,\Gamma(\beta-\gamma+1)}$$
$$+B.\frac{\Gamma(\gamma-\alpha-\beta+1)\,\Gamma(1-\gamma)}{\Gamma(1-\beta)\,\Gamma(1-\alpha)}$$

or $$1=\frac{(\pi \operatorname{cosec} \pi\gamma).\{\pi \operatorname{cosec} \pi\,(\gamma-\alpha-\beta)\}}{\{\pi \operatorname{cosec} \pi\,(\gamma-\alpha)\}\cdot\{\pi \operatorname{cosec} \pi\,(\gamma-\beta)\}}$$
$$+B\,\frac{\Gamma(\gamma-\alpha-\beta+1)\,\Gamma(1-\gamma)}{\Gamma(1-\beta)\,\Gamma(1-\alpha)}.$$

(Since $\Gamma(\alpha)\,\Gamma(1-\alpha)=\pi \operatorname{cosec} \pi\alpha$)

or $$B.\frac{\Gamma(\gamma-\alpha+\beta-1)\,(1-\gamma)}{\Gamma(1-\beta)\,\Gamma(1-\alpha)}=1-\frac{\sin \pi\,(\gamma-\alpha)\,\sin \pi\,(\gamma-\beta)}{\sin \pi\gamma\cdot\sin \pi\,(\gamma-\alpha-\beta)}$$
$$=\frac{\sin \pi\gamma\cdot\sin \pi\,(\gamma-\alpha-\beta)-\sin \pi\,(\gamma-\alpha)\cdot\sin \pi\,(\gamma-\beta)}{\sin \pi\gamma.\sin \pi.(\gamma-\alpha-\beta)}$$
$$=\frac{\cos \pi\,(\alpha+\beta)-\cos \pi\,(2\gamma-\alpha-\beta)-\cos \pi\,(\alpha-\beta)+\cos \pi\,(2\gamma-\alpha-\beta)}{2 \sin \pi\gamma.\sin \pi\,(\gamma-\alpha-\beta)}$$
$$=\frac{\cos \pi\,(\alpha+\beta)-\cos \pi\,(\alpha-\beta)}{2 \sin \pi\gamma.\sin \pi\,(\gamma-\alpha-\beta)}$$
$$=\frac{-2 \sin \pi\beta \sin \pi\alpha}{2 \sin \pi\gamma.\sin \pi\,(\gamma-\alpha-\beta)}$$

or $$B.\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}\times\frac{\{\Gamma(\alpha+\beta-\gamma)\,\Gamma(\gamma-\alpha-\beta+1)\}\,\{\Gamma(\gamma)\cdot\Gamma(1-\gamma)\}}{\{\Gamma(\beta)\,\Gamma(1-\beta)\}\,\{\Gamma(\alpha)\cdot\Gamma(1-\alpha)\}}$$
$$=\frac{\sin \pi\alpha.\sin \pi\beta}{\sin \pi\gamma.\sin \pi(\alpha+\beta-\gamma)}$$ or $$B\cdot\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma).\Gamma(\alpha+\beta-\gamma)}$$
$$\times\frac{\pi \operatorname{cosec} \pi\,(\alpha+\beta-\gamma)\}\,\{\pi \operatorname{cosec} \pi\gamma\}}{\{\pi \operatorname{cosec} \pi\beta\}\,\{\pi \operatorname{cosec} \pi\alpha\}}=\frac{\sin \pi\alpha\cdot\sin \pi\beta}{\sin \pi\gamma\,\sin \pi\,(\alpha+\beta-\gamma)}$$
$$\therefore\quad B=\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}.$$

Hence from (1), we have
$${}_2F_1\,(\alpha, \beta; \gamma; x)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma\{\gamma-\beta)}\cdot{}_2F_1\,(\alpha, \beta; \alpha+\beta-\gamma+1; 1-x)$$
$$+\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}.(1-x)^{\gamma-\alpha-\beta}$$
$$\times{}_2F_1\,(\gamma-\alpha, \gamma-\beta; \gamma-\alpha-\beta+1; 1-x).$$ [Meerut 73]

**§ 4·8. Symmetric Property of hypergeometric function.**

*Hypergeometric function does not change if the parameters α and β are interchanged, keeping γ fixed*

*i.e.* $${}_2F_1\,(\alpha, \beta, \gamma, x)={}_2F_1\,(\beta, \alpha, \gamma, x).$$

**§ 4·9. Integral formula for the hypergeometric function.**

*If* $|x| < 1$ *and if* $\gamma > \beta > 0$, *prove that*

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt.$$

**[Kanpur 84, 85, 86 ; Jodhpur 81, 83, 85 ; Agra 78, 81; Meerut 89]**

We have, the hypergeometric function

$${}_2F_1(\alpha, \beta, \gamma, x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!\,(\gamma)_n} x^n. \qquad \ldots\text{(i)}$$

where

$$(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$$

$$= \frac{1.2\ldots(\alpha-1)\,\alpha\,(\alpha+1)\,.\,(\alpha+n-1)}{1.2\ldots(\alpha-1)}$$

$$= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

$$\therefore \quad \frac{(\beta)_n}{(\gamma)_n} = \frac{\Gamma(\beta+n)}{\Gamma(\beta)} \cdot \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)}$$

$$= \frac{\Gamma(\gamma)\,\Gamma(\beta+n)\,\Gamma(\gamma-\beta)}{\Gamma(\beta)\,\Gamma(\gamma+n)\,\Gamma(\gamma-\beta)}.$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)} \frac{\Gamma(\beta+n)\,\Gamma(\gamma-\beta)}{\Gamma(\beta+n+\gamma-\beta)} \cdot \frac{1}{\Gamma(\gamma-\beta)}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \cdot \int_0^1 (1-t)^{\gamma-\beta-1}\,t^{\beta+n-1}\,dt$$

$$\left[\text{Since } \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = B(m, n) = \int_0^1 (1-t)^{n-1}\,t^{m-1}\,dt\right]$$

$$= \frac{1}{\dfrac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\beta+\gamma-\beta)}} \cdot \int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}\,dt$$

$$= \frac{1}{B(\beta, \gamma-\beta)} \cdot \int_0^1 t^{\beta+n-1}.(1-t)^{\gamma-\beta-1}\,dt$$

$\therefore$ From (i) we have

$${}_2F_1(\alpha, \beta, \gamma, x) = \sum_{n=0}^{\infty} \frac{1}{B(\beta, \gamma-\beta)} \cdot \int_0^1 t^{\beta+n-1}.(1-t)^{\gamma-\beta-1}\left(\frac{(\alpha)_n x^n}{n!}\right) dt$$

$$= \sum_{n=1}^{\infty} \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} . \frac{(\alpha_n).(xt)^n}{n!} . dt$$

$$=\frac{1}{B(\beta,\gamma-\beta)}\cdot\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}$$

$$\times\left[\sum_{n=0}^{\infty}\frac{(\alpha)_n.(xt)^n}{n!}\right]dt$$

$$=\frac{1}{B(\beta,\gamma-\beta)}\cdot\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}$$

$$\times\left\{1+\alpha.xt+\frac{\alpha(\alpha+1)}{2!}(xt)^2+\ldots\right\}dt$$

or $\quad {}_2F_1(\alpha,\beta,\gamma,x)=\dfrac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\cdot\displaystyle\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt$

which is known as the **integral formula** for hypergeometric function and is valid if

$$|x|<1,\ \gamma>\beta>0.$$

§ 4·10. **Kummer's Theorem.** Putting $x=-1$ and $\gamma=\beta-\alpha+1$ in the integral formula for hypergeometric function, we have

$${}_2F_1(\alpha,\beta,\beta-\alpha+1,-1)=\frac{1}{B(\beta,1-\alpha)}\cdot\int_0^1 t^{\beta-1}(1-t)^{-\alpha}(1+t)^{-\alpha}\,dt$$

$$=\frac{1}{\dfrac{\Gamma(\beta)\,\Gamma(1-\alpha)}{\Gamma(\beta-\alpha+1)}}\cdot\int_0^1 t^{\beta-1}(1-t^2)^{-\alpha}\,dt.$$

$$=\tfrac{1}{2}\cdot\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\,\Gamma(1-\alpha)}\int_0^1 z^{(\beta-2)/2}(1-z)^{-\alpha}.dt$$

putting $t^2=z$

$$=\tfrac{1}{2}\cdot\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\,\Gamma(1-\alpha)}\cdot\int z^{\beta/2-1}(1-z)^{(1-\alpha)-1}\,dz$$

$$=\tfrac{1}{2}\cdot\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\Gamma(1-\alpha)}.B(\beta/2,1-\alpha)$$

$$=\tfrac{1}{2}\cdot\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\,\Gamma(1-\alpha)}\cdot\frac{\Gamma(\beta/2)\,\Gamma(1-\alpha)}{\Gamma(\beta/2-\alpha+1)}$$

$$=\frac{\Gamma(\beta-\alpha+1).\beta/2.\Gamma(\beta/2)}{\alpha.\Gamma(\beta)\Gamma(\beta/2-\alpha+1)}$$

$$=\frac{\Gamma(\beta-\alpha+1)\,\Gamma(\beta/2+1)}{\Gamma(\beta+1)\,\Gamma(\beta/2+\alpha-1)}$$

Hence

$${}_2F_1(\alpha,\beta,\beta-\alpha+1,-1)=\frac{\Gamma(\beta-\alpha+1)\,\Gamma(\beta/2+1)}{\Gamma(\beta+1)\,\Gamma(\beta/2-\alpha+1)}\qquad\text{[Agra 80]}$$

which is known as **Kummer's theorem**,

## § 4·11. Gauss's Theorem.

[Meernt 76, 77, 90; Kanpur 71; B.H.U. 70, 72; Jodhpur 82, 84]

Putting $x=1$ in the integral formula for hypergeometric function, we have

$$
\begin{aligned}
{}_2F_1(\alpha, \beta, \gamma, 1) &= \frac{1}{B(\beta, \gamma-\beta)} \cdot \int_0^1 t^{\beta-1} \cdot (1-t)^{\gamma-\beta-1} \cdot (1-t)^{-\alpha}\, dt \\
&= \frac{1}{B(\beta, \gamma-\beta)} \cdot \int_0^1 t^{\beta-1} \cdot (1-t)^{(\gamma-\beta-\alpha)-1}\, dt \\
&= \frac{B(\beta, \gamma-\beta-\alpha)}{B(\beta, \gamma-\beta)} \\
&= \frac{\dfrac{\Gamma(\beta)\,\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)}}{\dfrac{\Gamma(\beta)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma)}} \qquad \because\ B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \\
&= \frac{\Gamma(\gamma)\,\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}
\end{aligned}
$$

Hence

$$
{}_2F_1(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}
$$

which is known as **Gauss's theorem.**

## § 4·12. Vandermonde's Theorem. [Jodhpur 82, Meerut 90]

Putting $\alpha=-n$ in Gauss's theorem, we have

$$
\begin{aligned}
{}_2F_1(-n, \beta, \gamma, 1) &= \frac{\Gamma(\gamma)\,\Gamma(\gamma-\beta+n)}{\Gamma(\gamma+n)\,\Gamma(\gamma-\beta)} \\
&= \frac{\Gamma(\gamma)\,(\gamma-\beta+n-1)\,(\gamma-\beta+n-2)\ldots(\gamma-\beta)\,\Gamma(\gamma-\beta)}{(\gamma+n-1)\,(\gamma+n-2)\ldots\gamma\,\Gamma(\gamma)\,\Gamma(\gamma-\beta)} \\
&= \frac{(\gamma-\beta+n-1)\,(\gamma-\beta+n-2)\quad(\gamma-\beta)}{(\gamma+n-1)\,(\gamma+n-2)\ldots(\gamma)} \\
&= \frac{(\gamma-\beta)\,(\gamma-\beta+1)\ldots(\gamma-\beta+n-1)}{\gamma\,(\gamma+1)\ldots(\gamma+n-1)}
\end{aligned}
$$

Hence

$$
\begin{aligned}
{}_2F_1(-n, \beta, \gamma, 1) &= \frac{(\gamma-\beta)\,(\gamma-\beta+1)\ldots(\gamma-\beta+n-1)}{\gamma\cdot(\gamma+1)\ldots(\gamma+n-1)} \\
&= \frac{(\gamma-\beta)_n}{(\gamma)_n}
\end{aligned}
$$

which is known as **Vandermonds theorem.**

## § 4·13. Differentiation of hypergeometric functions.

We have

$$
{}_2F_1(\alpha, \beta, \gamma, x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(\beta)_n}{n!\,(\gamma)_n}\, x^n.
$$

Differentiating both sides, w.r.t. '$x$', we have

$$\frac{d}{dx}\{{}_2F_1(\alpha, \beta, \gamma, x)\} = \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!\,(\gamma)_n}\, nx^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta_n)}{(n-1)!\,(\gamma)_n}\, x^{n-1}$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_{m+1} (\beta)_{m+1}}{m!\,(\gamma)_{m+1}}\, x^m. \text{ Putting } n-1=m$$

$$= \sum_{m=0}^{\infty} \frac{\alpha(\alpha+1)_m\, \beta(\beta+1)_m}{m!\,\gamma(\gamma+1)_m}\, x^m.$$

Since $(\alpha)_{m+1} = \alpha(\alpha+1)\ldots(\alpha+m+1-1)$

$= \{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+1+m-1)\}$

$= [\alpha(\alpha+1)_m]$

$$= \frac{\alpha\beta}{\gamma} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m (\beta+1)_m}{m!\,(\gamma+1)_m}\, x^m.$$

$$\therefore\quad \mathbf{\frac{d}{dx}\{{}_2F_1(\alpha, \beta, \gamma, x)\} = \frac{\alpha\beta}{\gamma}.\, {}_2F_1(\alpha+1, \beta+1, \gamma+1, x)}$$

**[Kanpur 81, 87; Meerut 74, 80 (S), 83 (P)]**

Differentiating again, w.r.t. '$x$', we have

$$\frac{d^2}{dx^2}\, {}_2F_1(\alpha, \beta, \gamma, x) = \frac{\alpha\beta}{\gamma}.\frac{d}{dx}\, {}_2F_1(\alpha+1, \beta+1, \gamma+1, x)$$

$$= \frac{\alpha\beta}{\gamma}.\frac{d}{dx} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m (\beta+1)_m}{m!\,(\gamma+1)_m}\, x^m$$

$$= \frac{\alpha\beta}{\gamma} \sum_{m=1}^{\infty} \frac{(\alpha+1)_m (\beta+1)_m}{(m-1)!\,(\gamma+1)_m}\, x^{m-1}$$

$$= \frac{\alpha\beta}{\gamma} \sum_{m=1}^{\infty} \frac{(\alpha+1)(\alpha+2)_{m-1} (\beta+1)(\beta+2)_{m-1}}{(m-1)!\,(\gamma+1)(\gamma+2)_{m-1}}\, x^{m-1}$$

$$= \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \sum_{n=0}^{\infty} \frac{(\alpha+2)_n (\beta+2)_n}{n!\,(\gamma+2)_n}$$

Putting $m-1=n$

$$= \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}.\, {}_2F_1(\alpha+2, \beta+2, \gamma+2, x)$$

Proceeding similarly, we have

$$\frac{d^p}{dx^p}\{{}_2F_1(\alpha,\beta,\gamma,x)\}=\frac{\alpha(\alpha+1)\ldots(\alpha+p-1)\,\beta(\beta+1)\ldots(\beta+p-1)}{\gamma(\gamma+1)\ldots(\gamma+p-1)}\,{}_2F_1(\alpha+p,\beta+p,\gamma+p,x)$$

$$=\frac{(\alpha)_p.(\beta)_p}{(\gamma)_p}.{}_2F_1(\alpha+p,\beta+p,\gamma+p,x).$$

**Particular case.** If $x=0$, we have

(i) $${}_2F_1(\alpha,\beta,\gamma,0)=\left[1+\frac{\alpha\beta}{\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\,\gamma(\gamma+1)}x^2+\ldots\right]_{x=0}$$

$$=1$$

(ii) $$\frac{d}{dx}\,{}_2F_1(\alpha,\beta,\gamma,0)=\frac{\alpha\beta}{\gamma}.{}_2F_1(\alpha+1,\beta+1,\gamma+1,0)$$

$$=\frac{\alpha\beta}{\gamma}.\left[1+\frac{(\alpha+1)(\beta+1)}{(\gamma+1)}x+\ldots\right]_{x=0}$$

$$=\frac{\alpha\beta}{\gamma}.$$

[Meerut 83 (P), 84 (P)]

(iii) $$\frac{d^2x}{dx^2}\,{}_2F_1(\alpha,\beta,\gamma,0)=\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}$$

(iv) $$\frac{d^p}{dx^p}\,{}_2F_1(\alpha,\beta,\gamma,0)=\frac{\alpha(\alpha+1)\ldots(\alpha+p-1)\,\beta(\beta+1)\ldots(\beta+p-1)}{\gamma(\gamma+1)\ldots(\gamma+p-1)}$$

$$=\frac{(\alpha)_p.(\beta)_p}{(\gamma)_p}.$$

## § 4·14. The confluent hypergeometric function.

The hypergeometric differential equation is

$$x(1-x)\frac{d^2y}{dx^2}+\{\gamma-(1+\alpha+\beta)x\}\frac{dy}{dx}-\alpha\beta y=0. \qquad \ldots(i)$$

Replacing $x$ by $\frac{x}{\beta}$, we have

$$\frac{x}{\beta}\left(1-\frac{x}{\beta}\right)\beta^2\frac{d^2y}{dx^2}+\left\{\gamma-(1+\alpha+\beta)\frac{x}{\beta}\right\}\beta\frac{dy}{dx}-\alpha\beta y=0$$

or $$x\left(1-\frac{x}{\beta}\right)\frac{d^2y}{dx^2}+\left\{\gamma-\left(1+\frac{1+\alpha}{\beta}\right)x\right\}\frac{dy}{dx}-\alpha y=0. \qquad \ldots(ii)$$

Since ${}_2F_1(\alpha,\beta;\gamma;x)$ is the solution of (i).

$\therefore\ {}_2F_1\left(\alpha,\beta;\gamma;\frac{x}{\beta}\right)$ is the solution of (ii).

Now if $\beta\to\infty$, we have

$$\lim_{\beta\to\infty}{}_2F_1\left(\alpha,\beta;\gamma;\frac{x}{\beta}\right)$$

is the solution of differential equation

$$x\frac{d^2y}{dx^2}+(\gamma-x)\frac{dy}{dx}-\alpha y=0. \qquad \ldots(iii)$$

Now

$$\lim_{\beta\to\infty} {}_2F_1\left(\alpha, \beta, \gamma; \frac{x}{\beta}\right) = \lim_{\beta\to\infty} \sum_{r=0}^{\infty} \frac{(\alpha)_r.(\beta)_r}{(\gamma)_r\, r\,!}\cdot\left(\frac{x}{\beta}\right)^r$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\gamma)_r}\cdot\frac{x^r}{r\,!} \qquad \left(\text{Since } \lim_{\beta\to\infty} \frac{(\beta)_r}{\beta^r}=1\right)$$

which is denoted by ${}_1F_1(\alpha; \gamma; x)$. This function is called **confluent hypergeometric function** and the equation (ii) is called **confluent hypergeometric function Kummer's equation.**

**Note.** The equation of the type (iii) occur in mathematical physics in the discussion of boundary value problems in potential theory.

§ 4·15. **Theorem.** *The confluent hypergeometric function ${}_1F_1(\alpha, \gamma, x)$ is a solution of the equation*

$$x\frac{d^2y}{dx^2}+(\gamma-x)\frac{dy}{dx}-\alpha y=0$$

(*the confluent hypergeometric equation of Kummer's equation*)

*If $\beta$ is not an integer a second independent solution is given by*

$$x^{1-\gamma}\,{}_1F_1(\alpha-\gamma+1, 2-\gamma, x)$$

*i.e. the general solution of the equation is*

$$y(x)=A\,{}_1F_1(\alpha, \gamma, x)+B\,x^{1-\gamma}\,{}_1F_1(\alpha-\gamma+1, 2-\gamma, x).$$

**Proof.** Proof ommitted to the reader as an exercise.

(Proceed as in § 4·6).

§ 4·16. **Whittaker's Confluent hypergeometric functions.**

The confluent hypergeometric equation is

$$x\frac{d^2y}{dx^2}+(\gamma-x)\frac{dy}{dx}-\alpha y=0 \qquad \text{...(i)}$$

whose general solution is

$$y=Ay_1(x)+By_2(x) \qquad \text{...(ii)}$$

where $A$ and $B$ are arbitrary constants and

$$y_1(x)={}_1F_1(\alpha, \gamma, x) \qquad \text{...(iii)}$$

and $$y_2(x)=x^{1-\gamma}\,{}_1F_1(\alpha-\gamma+1, 2-\gamma, x) \qquad \text{...(iv)}$$

Putting $$y(x)=x^{-\gamma/2}\,e^{\gamma/2}\,W(x) \qquad \text{...(v)}$$

in equation (i), we have

$$x\left[x^{-\gamma/2}e^{x/2}\frac{d^2W}{dx^2}+2\left(-\frac{\gamma}{2}x^{-(\gamma/2)-1}e^{x/2}+\tfrac{1}{2}x^{-\gamma/2}e^{x/2}\right)\frac{dW}{dx}\right.$$

$$\left.+W(x)\left\{\frac{\gamma}{2}\left(\frac{\gamma}{2}+1\right)x^{-(\gamma/2)-1}e^{x/2}-\frac{\gamma}{2}x^{-(\gamma/2)-1}.e^{x/2}+\tfrac{1}{4}x^{-\gamma/2}.e^{x/2}\right\}\right]$$

$$+(\gamma-x)\left[x^{-\gamma/2}e^{x/2}\frac{dW}{dx}+W(x)\left(-\frac{\gamma}{2}x^{-(\gamma/2)-1}+\tfrac{1}{2}x^{-\gamma/2}e^{x/2}\right)\right]$$

$$-\alpha\,x^{-\gamma/2}e^{x/2}\,W(x)=0$$

or $$\frac{d^2W}{dx^2}+\left[-\frac{1}{4}+\frac{\frac{\gamma}{2}-\alpha}{x}+\frac{\left(\frac{1}{4}-\frac{\gamma}{2}+\frac{\gamma^2}{4}\right)}{x^2}\right]W(x)=0$$

or $$\frac{d^2W}{dx^2}+\left[-\frac{1}{4}+\frac{k}{x}+\frac{\frac{1}{4}-m^2}{x^2}\right]W(x)=0 \qquad \text{...(vi)}$$

where $k=\frac{\gamma}{2}-\alpha$ and $m=\frac{1}{2}-\frac{\gamma}{2}$.

The solutions $W(x)$ of the equation (vi) are known as **Whittaker's Confluent hypergeometric functions.**

If $2m$ is neither 1 nor an integer the solution of the equation (i) corresponding to equation (vi) are given by (ii) with

$$\gamma=1+2m \text{ and } \alpha=\frac{\gamma}{2}-k=\tfrac{1}{2}+m-k.$$

Thus the solutions of equation (vi) are the Whittaker functions

$$M_{k,m}(x)=-x^{\gamma/2}\, e^{-x/2}\, y_1(x)$$
$$=x^{(1/2)+m}\, e^{-x/2}\, {}_1F_1\left(\tfrac{1}{2}-k+m,\, 1+2m,\, x\right)$$

and $$M_{k,-m}(x)=x^{(1/2)-m}\, e^{-x/2}\, {}_1F_1\left(\tfrac{1}{2}-k+m,\, 1-2m,\, x\right)$$

**§ 4·17. Integral representation of the confluent by hypergeometric function ${}_1F_1(a, \gamma, x)$.**

*Prove that*

$${}_1F_1(\alpha,\gamma,x)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}\int_0^1 (1-t)^{\gamma-\alpha-1}\, t^{\alpha-1}\, e^{xt}\, dt.$$

**[Meerut 76, 82, 85; Kanpur 71]**

**Proof.** We have

$${}_1F_1(\alpha,\gamma,x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n . x^n}{(\gamma)_n\, n\,!}. \qquad \text{...(i)}$$

where $$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

$$\therefore \quad \frac{(\alpha)_n}{(\gamma)_n}=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\cdot\frac{\Gamma(\gamma)}{\Gamma(\gamma+n)}$$
$$=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}\cdot\frac{\Gamma(\gamma-\alpha)\,\Gamma(\alpha+n)}{\Gamma(\beta+n+\gamma-\alpha)}$$
$$=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}\, B(\alpha+n,\, \gamma+\alpha)$$
$$=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}\int_0^1 (1-t)^{\gamma-\alpha-1} . t^{\alpha+n-1}\, dt$$

$\therefore$ from (1), we have

$$ {}_1F_1(\alpha, \gamma, x) = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)} \int_0^1 (1-t)^{\gamma-\alpha-1} . t^{\alpha+n-1} . \frac{x^n}{n!}\, dt $$

$$ = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)} \int_0^1 (1-t)^{\gamma-\alpha-1}\, t^{\alpha-1} \left\{ \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} \right\} dt $$

or $$ {}_1F_1(\alpha, \gamma, x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)} \int_0^1 (1-t)^{\gamma-\alpha-1}\, t^{\alpha-1}\, e^{xt}\, dt. $$

or $$ {}_1F_1(\alpha, \gamma, x) = \frac{1}{B(\alpha, \gamma-\alpha)} \int_0^1 (1-t)^{\gamma-\alpha-1}\, t^{\alpha-1}\, e^{xt}\, dt. $$

**Proved.**

**§ 4·18 Differentiation of confluent hypergeometric function.**

We have

$$ {}_1F_1(\alpha, \gamma, x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} . \frac{x^n}{n!} $$

Differentiating both sides w.r.t. '$x$' we have

$$ \frac{d}{dx} \{ {}_1F_1(\alpha, \gamma, x) \} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} . \frac{n x^{n-1}}{n!} $$

$$ = \sum_{n=1}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} . \frac{x^{n-1}}{(n-1)!} $$

$$ = \sum_{m=0}^{\infty} \frac{(\alpha)_{m+1}}{(\gamma)_{m+1}} . \frac{x^m}{m!} \qquad \text{Putting } n-1=m $$

$$ = \frac{\alpha}{\gamma} . \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{(\gamma+1)_m} . \frac{x^m}{m!} $$

$$ \{ \because \quad (\alpha)_{m+1} = \alpha . (\alpha+1)_m \} $$

$$ \therefore \quad \frac{\mathbf{d}}{\mathbf{dx}} \{ {}_1\mathbf{F}_1(\alpha, \gamma, \mathbf{x}) \} = \frac{\alpha}{\gamma}\, {}_1\mathbf{F}_1(\alpha+\mathbf{1}, \gamma+\mathbf{1}, \mathbf{x}). $$

Differentiating again, w.r.t. $x$, we have

$$ \frac{\mathbf{d^2}}{\mathbf{dx^2}} \{ {}_1\mathbf{F}_1(\alpha, \gamma, \mathbf{x}) \} = \frac{\alpha(\alpha+\mathbf{1})}{\gamma(\gamma+\mathbf{1})}\, {}_1\mathbf{F}_1(\alpha+\mathbf{2}, \gamma+\mathbf{2}, \mathbf{x}). $$

Proceeding similarly, we have

$$ \frac{\mathbf{d^p}}{\mathbf{dx^p}} \{ {}_1\mathbf{F}_1(\alpha, \gamma, \mathbf{x}) \} = \frac{(\alpha)_p}{(\gamma)_p}\, {}_1\mathbf{F}_1(\alpha+\mathbf{p}, \gamma+\mathbf{p}, \mathbf{x}) $$

**§ 4·19. Continuous hypergeometric functions. [Kanpur 84, 85]**

According to Gauss, *the function $F(a', b', c', z)$ is continuous to $F(a; b; c; z)$ if it is obtained by increasing or decreasing one and only one of the parameters $a, b, c$ by unity.*

Thus there are six hypergeometric function continuous to $F(a, b; c; z)$ and are denoted as

$$F(a+1, b; c; z)=F_{a+}, \quad F(a-1, b; c; z)=F_{a-}$$
$$F(a, b+1; c; z)=F_{b+}, \quad F(a, b-1; c; z)=F_{b-}$$
$$F(a, b; c+1; z)=F_{c+}, \quad F(a, b; c-1; z)=F_{c-}$$

§ 4·20. **Theorem.** *Between $F(a; b; c; z)$ and only two hypergeometric functions continuous to it, there exists a linear relation with polynomial coefficients.*

*i.e.* $(a-b)\, F=a\, F_{a+}-b\, F_{b+}$.

**[Meerut 85; Jodhpur 80, 82; Kanpur 84, 85]**

**Proof.** We have

$$(a-b)\ \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}\ F(a; b; c; z)$$

$$=(a-b).\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\cdot\frac{z^n}{n\,!}$$

$$=\sum_{n=0}^{\infty}\{(a+n)-(b+n)\}.\frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)}\cdot\frac{z^n}{n\,!}$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(a+n+1)\,\Gamma(b+n)}{\Gamma(c+n)}.\frac{z^n}{n\,!}$$

$$-\sum_{n=0}^{\infty}\frac{\Gamma(a+n).\Gamma(b+n+1)}{\Gamma(c+n)}\cdot\frac{z^n}{n\,!}$$

$$=\frac{\Gamma(a+1)\,\Gamma(b)}{\Gamma(c)}\sum_{n=0}^{\infty}\frac{(a+1)_n\,(b)_n}{(c)_n}\cdot\frac{z^n}{n\,!}$$

$$-\frac{\Gamma(a)\,\Gamma(b+1)}{\Gamma(c)}\sum_{n=0}^{\infty}\frac{(a)_n\,(b+1)_n}{(c)_n}\cdot\frac{z^n}{n\,!}$$

$$=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}\ a\ F(a+1, b; c; z)$$

$$-\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}.b\ F(a, b+1; c; z)$$

$$=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}.\{aF_{a+}-bF_{b+}\};$$

$\therefore$ $(a-b)\ F=aF_{a+}-bF_{b+}$

which holds when $|z|<1$. **Proved.**

Hence in general the theorem holds.

## § 4·21. Dixon's Theorem.

*Prove that*

$$ {}_3F_2\left[\begin{matrix}\alpha, & \beta, & \gamma & \\ 1+\alpha-\beta, & & 1+\alpha-\gamma & \end{matrix}\ 1\right] $$

$$ =\frac{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(1+\frac{\alpha}{2}-\beta-\gamma\right]\ \Gamma(1+\gamma-\beta)\ \Gamma(1+\alpha-\gamma)}{\Gamma(1+\gamma)\ (1+\gamma-\beta-\gamma)\ \Gamma\left(1+\frac{\alpha}{2}-\beta\right)\Gamma\left(1+\frac{\alpha}{2}-\gamma\right)} $$

[B.H.U. 70; Kanpur 83, 85]

**Proof.** We have

$$ \frac{\Gamma(\alpha)\ \Gamma(\beta)\ \Gamma(\gamma)}{\Gamma(1+\alpha-\beta)\ \Gamma(1+\alpha-\gamma)}\cdot {}_3F_2\left[\begin{matrix}\alpha, & \beta, & \gamma, & \\ 1+\alpha-\beta, & & 1+\alpha-\gamma & \end{matrix}\ 1\right] $$

$$ =\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\ \Gamma(\beta+n)\ \Gamma(\gamma+n)}{\Gamma(1+\alpha-\beta+n)\ \Gamma(1+\alpha-\gamma+n)}\cdot\frac{1}{n\,!} $$

$$ =\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\ \Gamma(\beta+n)\ \Gamma(\gamma+n)}{n\,!.\Gamma(1+\alpha+2n)\ \Gamma(1+\alpha-\beta-\gamma)} $$

$$ +\left\{\frac{\Gamma(1+\alpha+2n)\ \Gamma(1+\alpha-\beta-\gamma)}{\Gamma(1+\alpha-\beta+n).\Gamma(1+\alpha-\gamma+n)}\right\} $$

$$ =\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\ \Gamma(\beta+n)\ \Gamma(\gamma+n)}{n\,!\ \Gamma(1+\alpha+2n)\ \Gamma(1+\alpha-\beta-\gamma)} $$

$$ \times {}_2F_1(\beta+n,\ \gamma+n;\ 1+\alpha+2n;\ 1) $$

$$ \left\{\text{Since from Gauss's theorem § 4·11}\quad {}_2F_1(\alpha,\ \beta;\ \gamma;\ 1)=\frac{\Gamma(\gamma)\ \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\ \Gamma(\gamma-\beta)}\right\} $$

$$ =\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\ \Gamma(\beta+n)\ \Gamma(\gamma+n)}{n\,!\ \Gamma(1+\alpha+2n)\ \Gamma(1+\alpha-\beta-\gamma)} $$

$$ \left\{\sum_{m=0}^{\infty}\frac{(\beta+n)_m\ (\gamma+n)_m}{(1+\alpha+2n)_m}\cdot\frac{1}{m\,!}\right\} $$

$$ =\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha+n)\ \{\Gamma(\beta+n)\ (\beta+n)_m\}\ \{\Gamma(\gamma+n)\ (\gamma+n)_m\}}{\{\Gamma(1+\alpha+2n)\ (1+\alpha+2n)_m\}\ \Gamma(1+\alpha-\beta-\gamma)\ n\,!\ m\,!} $$

$$ =\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha+n)\ \Gamma(\beta+n+m)\ \Gamma(\gamma+n+m)}{\Gamma(1+\alpha+2n+m)\ \Gamma(1+\alpha-\beta-\gamma)\ n\,!\ m\,!} $$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(\beta+n+m)\,\Gamma(\gamma+n+m)\,\Gamma(\alpha+n)}{\Gamma(1+\alpha-\beta-\gamma)\,\Gamma(1+\alpha+2n+m)\,n\,!\,m\,!}$$

changing the order of summation)

Putting $p=n+m$

$$=\sum_{p=0}^{\infty}\frac{\Gamma(\beta+p)\,\Gamma(\gamma+p)}{\Gamma(1+\alpha-\beta-\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{n\,!\,(p-n)\,!\,\Gamma(1+\alpha+n+p)}$$

$$=\sum_{p=0}^{\infty}\frac{\Gamma(\beta+p)\,\Gamma(\gamma+p)}{\Gamma(1+\alpha-\beta-\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{n\,!\,\Gamma(1+\alpha+n+p)}$$

$$\times(-1)^n\frac{(-p)_n}{p\,!}$$

since $\dfrac{1}{(p+n)\,!}=(-1)^n\dfrac{(-p)_n}{p\,!}$

$$=\sum_{p=0}^{\infty}\frac{\Gamma(\beta+p)\,\Gamma(\gamma+p)\,\Gamma(\alpha)}{p\,!\,\Gamma(1+\alpha-\beta-\gamma).\Gamma(1+\alpha+p)}$$

$$\times\sum_{r=0}^{\infty}\frac{(\alpha)_n\,(-p)_n}{(1+\alpha+p)_n}\cdot\frac{(-1)^n}{n\,!}$$

$$=\sum_{p=0}^{\infty}\frac{\Gamma(\alpha)\,\Gamma(\beta+p)\,\Gamma(\gamma+p)}{p\,!\,\Gamma(1+\alpha-\beta-\gamma)\,\Gamma(1+\alpha+p)}$$

$$\times{}_2F_1\,(\alpha,\,-p;\,1+\alpha+p;\,-1)$$

$$=\sum_{p=0}^{\infty}\frac{\Gamma(\alpha)\,\Gamma(\beta+p)\,\Gamma(\gamma+p)}{p\,!\,\Gamma(1+\alpha-\beta-\gamma)\,\Gamma(1+\alpha+p)}$$

$$\times\frac{\Gamma(1+\alpha+p).\Gamma\left(\frac{\alpha}{2}+1\right)}{\Gamma(\alpha+1)\,\Gamma\left(\frac{\alpha}{2}+p+1\right)}$$

(from Kummer's theorem § 4·10)

$$=\sum_{p=0}^{\infty}\frac{\Gamma(\alpha)\,\Gamma(\beta+p)\,\Gamma(\gamma+1)\,\Gamma\left(\frac{\alpha}{2}+1\right)}{p\,!\,\Gamma(1+\alpha-\beta-\gamma)\,\Gamma(\alpha+1)\,\Gamma\left(\frac{\alpha}{2}+p+1\right)}$$

$$=\frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)}{\Gamma(1+\alpha)\,\Gamma(1+\alpha-\beta-\gamma)}\sum_{p=0}^{\infty}\frac{(\beta)_p\,(\gamma)_p}{\left(\frac{\alpha}{2}+1\right)_p}\cdot\frac{1}{p\,!}$$

$$=\frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)}{\Gamma(1+\alpha)\,\Gamma(1+\alpha-\beta-\gamma)}\cdot {}_2F_1\left(\beta;\gamma;\frac{\alpha}{2}+1;1\right)$$

$$=\frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)}{\Gamma(1+\alpha)\,\Gamma(1+\alpha-\beta-\gamma)}\cdot\frac{\Gamma\left(\frac{\alpha}{2}+1\right)\Gamma\left(\frac{\alpha}{2}+1-\beta-\gamma\right)}{\Gamma\left(\frac{\alpha}{2}+1-\beta\right)\Gamma\left(\frac{\alpha}{2}+1-\gamma\right)}$$

(from Gauss's theorem § 4·11)

**Hence**

$${}_3F_2\left[\begin{matrix}\alpha, & \beta, & \gamma; & 1\\ 1+\alpha-\beta, & 1+\alpha-\gamma & & \end{matrix}\right]$$

$$=\frac{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(1+\frac{\alpha}{2}-\beta-\gamma\right)\Gamma(1+\alpha-\beta)\,\Gamma(1+\alpha-\gamma)}{\Gamma(1+\alpha)\,\Gamma(1+\alpha-\beta-\gamma)\,\Gamma\left(1+\frac{\alpha}{2}-\beta\right)\Gamma\left(1+\frac{\alpha}{2}-\gamma\right)}$$

**Proved.**

**Examples.**

**Ex. 1.** *If* $|z|<1$ *and* $\left|\frac{z}{1-z}\right|<1$

*prove that*

$${}_2F_1(a, b, c, z)=(1-z)^{-a}\,{}_2F_1\left(a, c-b, c, -\frac{z}{1-z}\right)$$

**[Jodhpur 80, 82; Agra 79; Meerut 71, 77 (S), 82 (P); Kanpur 71, 83, 84]**

**Sol.** From integral formula for the hyper-geometric function we have

$${}_2F_1(a, b, c, z)=\frac{1}{B(b, c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt$$

$$=\frac{1}{B(b, c-b)}\int_0^1 (1-v)^{b-1}\,v^{c-b-1}\,(1-z+vz)^{-a}\,dv$$

Putting $t=1-v$

so that $dt=-dv$

$$=\frac{1}{B(b, c-b)}\int_0^1 v^{(c-b)-1}\,(1-v)^{c-(c-b)-1}\times(1-z)^{-a}\left(1-\frac{z}{-1+z}\,v\right)^{-a}dt$$

$$=(1-z)^{-a}\cdot\frac{1}{B(b, c-b)}\int_0^1 v^{(c-b)-1}\,(1-v)^{c-(c-b)-1}\times\left(1-\frac{z}{-1+z}\right)^{-a}dt$$

$$=(1-z)^{-a}\ {}_2F_1\left(a,\ c-b,\ c,\ \frac{z}{-1+z}\right)$$

$$=(1-z)^{-a}\ {}_2F_1\left(a,\ c-b,\ c,\ -\frac{z}{1-z}\right).$$

**Proved.**

**Ex. 2.** *Show that*

$$P_n\ (\cos\theta)=\cos\theta\ {}_2F_1\left(-\frac{n}{2},\ -\frac{n-1}{2},\ 1,\ -\tan^2\theta\right).$$

**[Kanpur 83; Meerut 89]**

**Sol.** From Laplace's first integral, we have

$$P_n(x)=\frac{1}{\pi}\int_0^\pi [x+\sqrt{(x^2-1)}\cos\phi]^n\ d\phi.$$

Putting $x=\cos\theta$, we have

$$P_n\ (\cos\theta)=\frac{1}{\pi}\int_0^\pi [\cos\theta+\sqrt{(\cos^2\theta-1)}\cos\phi]^n\ d\phi$$

$$=\frac{1}{\pi}\int_0^\pi [\cos\theta+i\sin\theta\cos\phi]^n\ d\phi$$

$$=\frac{1}{\pi}\cos^n\theta\int_0^\pi [1+i\tan\theta\cos\phi]^n\ d\phi$$

$$=\frac{1}{\pi}\cos^n\theta\int_0^\pi \Big[1+ni\tan\theta\cos\phi$$

$$+\frac{n(n-1)}{2!}i^2\tan^2\theta\cos^2\phi$$

$$+\frac{n(n-1)(n-2)}{3!}i^3\tan^3\theta\cos^3\phi+\ldots\Big]\ d\phi$$

$$=\frac{1}{\pi}\cos^n\theta\Big[\int_0^\pi d\phi+0+2.\frac{n(n-1)}{2!}\times$$

$$i^2\tan^2\theta\int_0^{\pi/2}\cos^2\phi\ d\phi+0\ldots$$

By the property of definite integrals

$$=\frac{1}{\pi}\cos^n\theta\Big[\pi+i^2\frac{n(n-1)}{1!}.\tan^2\theta.\frac{\pi}{4}$$

$$+i^4.\frac{n(n-1)(n-2)(n-3)}{4!}.2.\frac{1.3}{2.4}\frac{\pi}{4}-\ldots\Big]$$

$$=\cos^n\theta\ [1+\frac{\left(-\frac{n}{2}\right)\left(-\frac{n-1}{2}\right)}{1!}(-\tan^2\theta)$$

$$+\frac{\left(-\frac{n}{2}\right)\left(-\frac{n}{2}+1\right)\cdot\left(-\frac{n-1}{2}\right)\left(-\frac{n-1}{2}+1\right)}{2.1.2!}.$$

$$(-\tan^2\theta)^2+\ldots]$$

$$=\cos^n\theta\ {}_2F_1\left(-\frac{n}{2}, -\frac{n-1}{2}, 1, -\tan^2\theta\right).$$

**Proved.**

**Ex. 3.** *Prove that*

$$P_n(x)={}_2F_1\left(-n, n+1, 1, \frac{1-x}{2}\right).$$

**[Jodhpur 84; Raj. 86; Meerut 73(S), 75, 83, 90; B.H.U. 72; Agra 82, 85; Kanpur 84, 85, 86, 87]**

**Sol.** From Rodrigue's formula we have

$$P_n(x)=\frac{1}{2^n\, n\,!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$=\frac{1}{n\,!}\frac{d^n}{dx^n}[(x-1)^n\{\tfrac{1}{2}(x+1)\}^n]$$

$$=\frac{1}{n\,!}\frac{d^n}{dx^n}[(x-1)^n\{1-\tfrac{1}{2}(1-x)\}^n]$$

$$=\frac{(-1)^n}{n\,!}\frac{d^n}{dx^n}\left[(1-x)^n.\left\{1-n.\frac{1}{2}(1-x)+\frac{n(n-1)}{2\,!}.\frac{(1-x)^2}{4}-\frac{n(n-1)(n-2)}{3\,!}.\frac{(1-x)^3}{8}\cdots\right\}\right]$$

$$=\frac{(-1)^n}{n\,!}\frac{d^n}{dx^n}\left[(1-x)^n-\frac{n}{2}(1-x)^{n+1}+\frac{n(n-1)}{2\,!\,2^2}(1-x)^{n+2}-\frac{n(n-1)(n-2)}{3\,!\,2^3}(1-x)^{n+3}\cdots\right]$$

$$=\frac{(-1)^n}{n\,!}\left[(-1)^n n\,!-\frac{n}{2}(-1)^n\frac{(n+1)\,!}{1\,!}(1-x)+\frac{n(n-1)}{2\,!}.\frac{(-1)^n}{2^2}\frac{(n+2)\,!}{2\,!}(1-x)^2-\cdots\right]$$

$$=1+\frac{(-n).(n+1)}{1.1\,!}\left(\frac{1-x}{2}\right)+\frac{(-n)(-n+1).(n+1)(n+2)}{2.1.2\,!}\left(\frac{1-x}{2}\right)^2+\cdots$$

$$={}_2F_1\left(-n, n+1; 1, \frac{1-x}{2}\right)$$

**Proved.**

**Ex. 4.** *Prove that*

*(i)* $P_n(\cos\theta)={}_2F_1\left(-n, n+1; 1; \sin^2\frac{\theta}{2}\right)$. **[Agra 83, 86]**

*and (ii)* $P_n(\cos\theta)=(-1)^n\ {}_2F_1\left(n+1, -n; 1; \cos^2\frac{\theta}{2}\right)$.

**Sol.** From the last Ex. we have

$$P_n(x)=\left(-n, n+1; 1: \frac{1-x}{2}\right). \quad \ldots(1)$$

Putting $x=\cos\theta$ in (1), we have

$$P_n(\cos\theta)={}_2F_1\left(-n, n+1; 1; \sin^2\frac{\theta}{2}\right)$$

**Proved.**

(ii) Putting $x=-\cos\theta$ in (1), we have

$$P_n(-\cos\theta)={}_2F_1\left(-n, n+1; 1; \cos^2\frac{\theta}{2}\right)$$

or $$(-1)^n P_n(\cos\theta)={}_2F_1\left(-n, n+1; 1; \cos^2\frac{\theta}{2}\right)$$

or $$P_n(\cos\theta)=(-1)^n\,{}_2F_1\left(-n, n+1; 1; \cos^2\frac{\theta}{2}\right)$$

$$\therefore\quad P_n(-x)=(-1)^n P_n(x)$$

or $$P_n(\cos\theta)=(-1)^n\,{}_2F_1\left(n+1, -n; 1; \cos^2\frac{\theta}{2}\right)$$

(Since in hyper-geometric series $\alpha$ and $\beta$ can be interchanged).

**Proved.**

**Ex. 5.** *Prove that*

$$F\left(\alpha, \beta; \gamma; \frac{1}{x}\right)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,x^{\alpha}\,{}_2F_1(\alpha, \alpha-\gamma+1;$$
$$\alpha+\beta-\gamma+1; 1-x)+\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}$$
$$\times x^{\beta}(x-1)^{\gamma-\alpha-\beta}$$
$${}_2F_1(\gamma-\alpha, 1-\alpha; \gamma-\alpha-\beta+1; 1-x)$$

*where* $1 < x < 2$ and $1 > \gamma > \alpha+\beta$.

**Sol.** From § 4·7, we have

$${}_2F_1(\alpha, \beta; \gamma; x)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,{}_2F_1(\alpha, \beta; \alpha+\beta-\gamma+1, 1-x)$$
$$+\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}(1-x)^{\gamma-\alpha+\beta}$$
$${}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma-\alpha-\beta+1; 1-\alpha)$$

Replacing $x$ by $\frac{1}{x}$, we have

$${}_2F_1\left(\alpha, \beta; \gamma; \frac{1}{x}\right)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,{}_2F_1\left(\alpha, \beta; \alpha+\beta-\gamma+1; 1-\frac{1}{x}\right)$$
$$+\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta}$$
$${}_2F_1\left(\gamma-\alpha, \gamma-\beta; \gamma-\alpha-\beta+1; 1-\frac{1}{x}\right) \quad \text{...(i)}$$

Now from Ex. 1, we have

$${}_2F_1(\alpha, \beta; \gamma; x)=(1-x)^{-\alpha}\,{}_2F_1\left(\alpha, \gamma-\beta; \gamma; \frac{1}{x}\right) \quad \text{...(ii)}$$

Replacing $\gamma$ by $\alpha+\beta-\gamma+1$, and $x$ by $1-\frac{1}{x}$, we have

$$ {}_2F_1\left(\alpha, \beta; \alpha+\beta-\gamma+1, 1-\frac{1}{x}\right) $$
$$ = x^{\alpha}\ {}_2F_1(\alpha, \alpha-\gamma+1; \alpha+\beta-\gamma+1; 1-x). \qquad \text{...(iii)} $$

Again replacing $\alpha$ by $\gamma-\alpha$, $\beta$ by $\gamma-\beta$, $\gamma$ by $\gamma-\alpha-\beta+1$ and $x$ by $1-\frac{1}{x}$ in (ii), we have

$$ {}_2F_1\left(\gamma-\alpha, \gamma-\beta; \gamma-\alpha-\beta+1; 1-\frac{1}{x}\right) $$
$$ = x^{\gamma-\alpha}\ {}_2F_1(\gamma-\alpha; 1-\alpha; \gamma-\alpha-\beta+1; 1-x). \qquad \text{...(iv)} $$

Hence from (i), (iii) and (iv), we have

$$ {}_2F_1\left(\alpha, \beta; y; \frac{1}{x}\right) $$
$$ = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}.x^{\alpha}\ {}_2F_1(\alpha, \alpha-\gamma+1; \alpha+\beta-\gamma+1; 1-x) $$
$$ +\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}.\left(1-\frac{1}{x}\right)^{\gamma-\alpha-\beta} x^{\gamma-\alpha} $$
$$ \times\,{}_2F_1(\alpha, \alpha-\gamma+1; \alpha+\beta-\gamma+1; 1-x) $$
$$ = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}.x^{\alpha}\ {}_2F_1(\alpha, \alpha-\gamma+1; \alpha+\beta-\gamma+1; 1-x) $$
$$ +\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}.x^{\beta}.(x-1)^{\gamma-\alpha-\beta} $$
$$ {}_2F_1(\alpha, \alpha-\gamma+1; \alpha+\beta-\gamma+1; 1-x) $$

**Proved.**

**Ex. 6.** *Prove that*

$$ {}_1F_1(\alpha, \gamma, x) = e^{x}\ {}_1F_1(\gamma-\alpha, \gamma, -x). $$ **[Kanpur 86]**

**Sol.** $$ {}_1F_1(\alpha, \gamma, x) = \frac{1}{B(\alpha, \gamma-\alpha)}\int_0^1 (1-t)^{\gamma-\alpha-1}\,t^{\alpha-1}\,e^{xt}\,dt \qquad \text{...(i)} $$

Replacing $x$ by $-x$ and $\alpha$ by $\gamma-\alpha$, we have

$$ {}_1F_1(\gamma-\alpha, \gamma, -x) = \frac{1}{B(\gamma-\alpha, \alpha)}\int_0^1 (1-t)^{\alpha-1}\,t^{\gamma-\alpha-1}\,e^{-t}\,dt $$
$$ = \frac{e^{-x}}{B(\alpha, \gamma-\alpha)}\int_0^1 (1-t)^{\alpha-1}\,t^{\gamma-\alpha-1}\,e^{x(1-t)}\,dt $$

Putting $1-t=u$ so that $dt=-du$

$$ = \frac{-e^{-x}}{B(\alpha, \gamma-\alpha)}\int_0^1 u^{\alpha-1}\,(1-u)^{\gamma-\alpha-1}\,e^{ux}\,du $$

or $$ e^{x}\ {}_1F_1(\gamma-\alpha; \gamma, -x) = \frac{1}{B(\alpha, \gamma-\alpha)}\int_0^1 t^{\alpha-1}\,(1-t)^{\gamma-\alpha-1}\,e^{tx}\,dt $$

or $$ e^{x}\ {}_1F_1(\gamma-\alpha, \gamma, -x) = {}_1F_1(\alpha, \gamma, x). $$

**Ex. 7.** *Prove that*

$$ (a-c+1)\,F = a\,F_{a+} - (c-1)\,F_{c-} $$ **[Jodhpur 80, 82]**

**Sol.** $(a-c+1)\dfrac{\Gamma(a)\,(Tb)}{\Gamma(c)}\,F(a, b; c; z)$

$$=(a-c+1).\frac{\Gamma(a)\,\Gamma(b)}{T(c)}\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n\,z^n}{(c)_n\,n\,!}$$

$$=\sum_{n=0}^{\infty}\{(a+n)-(c-1+n)\}\frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)}\cdot\frac{z^n}{n\,!}$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(a+n+1)\,\Gamma(b+n)}{\Gamma(c+n)}\cdot\frac{z^n}{n\,!}$$

$$-\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n-1)}\cdot\frac{z^n}{n\,!}$$

$$=\frac{\Gamma(a+1)\,\Gamma(b)}{T(c)}\cdot\sum_{n=0}^{\infty}\frac{(a+1)_n\,(b)_n}{(c)_n}\cdot\frac{z^n}{n\,!}$$

$$-\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c-1)}\cdot\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c-1)_n}\cdot\frac{z^n}{n\,!}$$

$$=\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}.a\,F(a+1, b; c; z)$$

$$-\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}(c-1)\,F(a, b; c-1; z)$$

$$=\frac{\Gamma(a)\,\Gamma(b)}{T(c)}.\{aF_{a+}-(c-1)\,F_{c-}\}$$

$\therefore\quad (a-c+1)\,F=aF_{a+}-(c-1)\,F_{c-}$ **Proved.**

## Exercise on Chapter IV

**1.** Show that the function $\omega={}_0F_1\,(\rho;\,z)$ satisfies the equation

$$z\,\frac{d^2\omega}{dz^2}+\rho\,\frac{d\omega}{dz}-\omega=0.$$

Hence show that

$${}_1F_1\,(\alpha;\,2\alpha;\,2z)=e^z\,{}_0F_1\left(\alpha,\,\alpha+\frac{1}{2},\,\frac{z^2}{4}\right)$$

Provided that $2\alpha$ is not a negative integer or zero. [B.H U. 72]

**2.** If $P_n(x)$ be Legendre's Polynomial of degree $n$ prove that

$$P_n(x)=\left(\frac{x-1}{2}\right)^n{}_2F_1\left(-n,\,-n;\,1,\,\frac{x+1}{x-1}\right)$$

**3.** Prove Ramanujan's formula

$${}_1F_1\,(\alpha;\,\beta;\,z)\,{}_1F_1\,(\alpha;\,\beta;\,-z)={}_2F_3\begin{bmatrix}\alpha,\,\beta-\alpha; & \dfrac{z^2}{4}\\ \beta,\,\dfrac{\beta}{2},\,\dfrac{\beta}{2}+\dfrac{1}{2} & \end{bmatrix}$$

4. Prove that

$$[F(a, b; a+b+\tfrac{1}{2}; z]^2 = {}_3F_2 \left[ \begin{matrix} 2a,\ a+b,\ 2b;\ z \\ a+b+\tfrac{1}{2},\ a+2b \end{matrix} \right]$$

5. Show that

(i) $(1-x)^{-2} = {}_2F_1\,(\alpha, \beta; \beta; x)$ [Meerut 78]

(ii) $e^x = {}_1F_1\,(\alpha; \alpha; x)$. [Meerut 78]

6. Show that

$$\int_0^{\pi/2} (1-k^2 \sin^2 \phi)^{-1/2}\, d\phi = \frac{\pi}{2} F\,(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2)$$ [Meerut 79]

where $|k| < 1$.

7. Show that $\log \dfrac{1+x}{1-x} = 2x\, F\,(\tfrac{1}{2}, 1, \tfrac{3}{2}; x^2)$. [Meerut 84]

8. ${}_2F_1\,(\alpha, \beta: \gamma; \tfrac{1}{2}) = 2^{\alpha}\, {}_2F_1\,(\alpha, \gamma-\beta; \gamma; -1)$. [Meerut 84 (P)]

9. ${}_2F_1\,(\alpha, \beta; \tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2}; \tfrac{1}{2}) = \dfrac{\Gamma(\tfrac{1}{2})\, \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta)}{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\alpha)\, \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\beta)}$

[Meerut 84 (P(, 87]

10. Show that

$$F\,(\alpha, 1-\alpha; \gamma; \tfrac{1}{2}) = \frac{\Gamma(\tfrac{1}{2}\gamma)\, \Gamma(\tfrac{1}{2}\gamma+\tfrac{1}{2})}{\Gamma(\tfrac{1}{2}\alpha+\tfrac{1}{2}\gamma)\, \Gamma(\tfrac{1}{2}-\tfrac{1}{2}\alpha+\tfrac{1}{2}\gamma)}$$ [Meerut 86]

11. Prove that

$$F(\alpha, \beta+1; \gamma+1; x) - F\,(\alpha, \beta; \gamma; x) = \frac{\alpha\,(\gamma-\beta)}{\gamma\,(\gamma+1)}\, x\, F\,(\alpha+1; \beta+1; \gamma+2; x).$$ [Meerut 88]

12. $[\alpha+(\beta-\gamma)\, z]\, {}_2F_1\,(\alpha, \beta; \gamma; z) = \alpha\,(1-z)\, {}_2F_1\,(\alpha+1, \beta; \gamma; z)$
$-\gamma^{-1}\,(\gamma-\alpha)\,(\gamma-\beta)\, z\, {}_2F_1\,(\alpha, \beta; \gamma+1; z)$ [Jodhpur 81, 85]

13. $(1-z)\, {}_2F_1\,(\alpha, \beta; \gamma; z) = {}_2F_1\,(\alpha-1, \beta; \gamma; z)$
$-\gamma^{-1}\,(\gamma-\beta)\, z\, {}_2F_1\,(\alpha, \beta; \gamma+1; z)$ [Jodhpur 81, 85]

# 5
# Bessel's Equations

### § 5·1. Bessel's Equation (Def.)

The differential equation of the form

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0$$

is called **Bessel's differential equation (or Bes el's equation).**

### § 5·2. Solution of Bessel's differential equation.

**[Indore 79; Agra 83; Meerut M.Sc Phy. 79; Meerut 85; Kanpur 88; Jodhpur 81]**

The Bessel's differential equation is

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0. \qquad (1)$$

We can integrate (1) in a series of ascanding powers of $x$. Let us assume that its series solution is

$$y=\sum_{r=0}^{\infty} a_r\, x^{k+r}$$

$$\therefore \quad \frac{dy}{dx}=\sum_{r=0}^{\infty} a_r\,(k+r)\,x^{k+r-1}$$

and
$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r\,(k+r)\,(k+r-1)\,x^{k+r-2}$$

Substituting these values in (1) we have

$$\sum_{r=0}^{\infty} a_r\left[(k+r)\,(k+r-1)\,x^{k+r-2}+\frac{1}{x}\,(k+r)\,x^{k+r-1}+\left(1-\frac{n^2}{x^2}\right)x^{k+r}\right]=0$$

or
$$\sum_{r=0}^{\infty} [a_r\,\{(k+r)^2-n^2\}\,x^{k+r-2}+x^{k+r}]=0 \qquad ...(2)$$

Since relation (2) is an identity, the coefficients of various powers of $x$ must be zero.

$\therefore$ Equating to zero the coefficient of lowest power of $x$, *i.e.* of $x^{n-2}$ in (2), we have

$$a_0\,(k^2-n^2)=0.$$

Now $a_0 \neq 0$ as it is the coefficient of the first term with which we begin the series.

$$\therefore\quad k^2-n^2=0$$

or $$k=\pm n. \qquad \ldots(3)$$

Now equating to zero the coefficient of $x^{k-1}$ in (2), we have

$$a_1\,\{(k+1)^2-n^2\}=0.$$

But $(k+1)^2-n^2 \neq 0$ for $k=\pm n$ given by (3)

$$\therefore\quad a_1=0.$$

Again equating to zero the coefficient of general term *i.e.* of $x^{k+r}$ in (2), we have

$$a_{r+2}\,\{(k+r+2)^2-n^2\}+a_r=0$$

or $$a_{r+2}\,(k+r+n+2)\,(k+r-n+2)=-a_r.$$

$$\therefore\quad a_{r+2}=-\frac{a_r}{(k+r+n+2)\,(k+r-n+2)} \qquad \ldots(4)$$

Putting $r=1$ in (4). we have

$$a_3=\frac{a_1}{(k+n+3)\,(k-n+3)}$$

$$=0 \qquad \text{since } a_1=0.$$

Similarly putting $r=3$, $5$...etc. in (4), we have

$$a_1=a_3=a_5\ldots=0 \text{ (each)}.$$

Now two cases arise.

**Case I.** When $k=n$, from (4), we have

$$a_{r+2}=-\frac{a_r}{(2n+r+2)\,(r+2)}$$

Putting $r=0, 2, 4$ etc.

$$a_2=-\frac{a_0}{(2n+2)\,(2)}$$

$$=-\frac{a_0}{2^2.1\,!\,(n+1)}$$

$$a_4=-\frac{a_2}{(2n+4)\,(4)}=-\frac{a_2}{2^2.2\,!\,(n+2)}$$

$$=\frac{a_0}{2^4.2\,!\,(n+1)\,(n+2)} \text{ etc.}$$

$$\therefore\quad y=a_0\left[x^n-\frac{x^{n+2}}{2^2.1\,!\,(n+1)}+\frac{x^{n+4}}{2^4\,2\,!\,(n+1)\,(n+2)}\cdots\right]$$

$$=a_0 x^n.\left[1+(-1)\frac{x^2}{2^2.1\,!\,(n+1)}+(-1)^2\frac{x^4}{2^4\,2\,!\,(n+1)\,(n+2)}+\ldots\right]$$

If $a_0=\dfrac{1}{2^n\,\Gamma(n+1)}$, this solution is called $J_n(x)$, known as **Bessel's function of the first kind of order $n$.** [Meerut 85]

$$\therefore\quad J_n(x)=\frac{x^n}{2^n\,\Gamma(n+1)}\left[1+(-1)\frac{x^2}{2^2.1\,!\,(n+1)}+(-1)^2\frac{x^4}{2^4.2\,!\,(n+1)\,(n+2)}+\ldots\right]$$

$$=\frac{x^n}{2^n\,\Gamma(n+1)}\times\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r}}{2^r\,r\,!\,(n+1)\,(n+2)\ldots(n+r)}$$

$$=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r\,!\,\Gamma(n+r+1)}$$ [Meerut 73 (S)]

**Case II.** When $k=-n$.

The series solution is obtained by replacing $n$ by $-n$ in the value of $J_n$.

$$\therefore\quad J_{-n}(x)=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{-n+2r}\frac{1}{r\,!\,\Gamma(-n+r+1)}.$$

When $n$ is not integer $J_{-n}(x)$ is distinct from $J_n(x)$.

**Note.** The solution of Bessel's equation are called Bessel's functions.

**§ 5·3. General solution of Bessel's Equation.**

The most general solution of Bessel's equation is

$$y=A\,J_{-n}(x)+B\,J_n(x)$$

where $A$, $B$ are two arbitrary constants.

**§ 5·4. Integration of Bessel's equation in series for n=0.** [Kanpur 83]

The differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0. \qquad \ldots(1)$$

is known as Bessel's equation for $n=0$.

Let us assume that its solution is

$$y=\sum_{r=0}^{\infty} a_r\, x^{k+r}$$

$$\therefore \quad \frac{dy}{dx}=\sum_{n=0}^{\infty} a_r\,(k+r)\, x^{k+r-1}$$

and
$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r\,(k+r)\,(k+r-1)\, x^{r+r-2}$$

Substituting these values in (1), we get

$$\sum_{r=0}^{\infty} a_r\left[(k+r)(k+r-1)\, x^{k+r-2}+\frac{1}{x}\,(k+r)\, x^{k+r-1}+x^{k+r}\right]=0$$

or
$$\sum_{r=0}^{\infty} a_r\,[(k+r)^2\, x^{k+r-2}+x^{k+r}]=0 \qquad \ldots(2)$$

which is an identity. Equating to zero the coefficient of lowest power of $x$ *i.e.* $x^{k-2}$, we have

$a_0k^2=0$. Since $a_0\neq 0$ for the same reason, as in § 5·2

$$k^2=0. \qquad \therefore \quad k=0. \qquad \ldots(3)$$

Now equating to zero the coefficient of next power of $x$, *i.e.*, $x^{k+1}$, we have

$$a_1\,(k+1)^2=0.$$

Since $k+1\neq 0$ by virtue of (3), we have

$$a_1=0.$$

Again equating to zero the coefficient of the general term, *i.e.* $x^{k-r}$, we have

$$a_{r+2}\,(k+r+2)^2+a_r=0$$

$$\therefore \quad a_{r+2}=-\frac{a_r}{(k+r+2)^2}.$$

When $k=0$.

$$a_{r+2}=-\frac{a_r}{(r+2)^2}.$$

Putting $r=1, 3, 5, \ldots$ etc.,

we have $a_1=a_3=a_5=\ldots=0$ (each).

Again putting $r=0, 2, 4,$ etc., we have

$$a_2=-\frac{a_0}{2^2}$$

$$a_4=-\frac{a_2}{4^2}=\frac{a_0}{2^2.4^2} \text{ etc.}$$

Since $y=\sum_{r=0}^{\infty} a_r\, x^r$, when $k=0$

$$\therefore\ y=a_0\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}\cdots\right)$$

If $a_0=1$, this solution is denoted by $J_0(x)$

$$\therefore\ \mathbf{J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\cdots}$$ **[Kanpur 85]**

**where $J_0(x)$ is called Bessel's function of zeroeth order.**

**§ 5·5. Definition of $J_0(x)$.**

***$J_0(x)$ is that solution of Bessel's equation for $n=0$ i.e. of***

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0$$

***which is equal to 1 for $x=0$***

**§ 5·6. Recurrence formulae $J_n(x)$.**

**(I) $x.J'_n(x)=n\cdot J_n(x)-x.J_{n+1}(x)$.**

**[Agra 79, 86 ; Meerut 79 (S), 81 ; Raj. 78 ; Rohilkhand 80, 83, 88 ; Jodhpur 83, 86 ; Kanpur 80]**

**Proof.** We know that

$$J_n(x)=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!\,\Gamma(n+r+1)}$$

where $n$ is a positive integer.

Differentitating w.r.t. $x$, we have

$$J'_n(x)=\sum_{r=0}^{\infty}(-1)^r\frac{(n+2r)}{r!\,\Gamma(n+r+1)}\frac{1}{2}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$\therefore\ xJ'_n(x)=\sum_{r=0}^{\infty}(-1)^r\frac{(n+2r)}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=\sum_{r=0}^{\infty}(-1)^r\frac{n}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$+\sum_{r=0}^{\infty}(-1)^r\frac{2r}{r!.\Gamma(n+r+1)}.\frac{x}{2}.\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=nJ_n(x)+x\sum_{r=1}^{\infty}(-1)^r\frac{1}{(r-1)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

Putting $r-1=s$

$$=nJ_n(x)-x\sum_{s=0}^{\infty}(-1)^s\frac{1}{s!.\Gamma(n+1-s+1)}\left(\frac{x}{2}\right)^{\overline{n+1}+2s}$$

$$=nJ_n(x)-xJ_{n+1}(x).$$

Hence $\quad xJ'_n = nJ_n(x) \quad xJ_{n+1}(x).$

which may also be written as

$$\frac{d}{dx}(x^{-n} J_n) = \quad x^{-n} J_{n+1}.$$

**(II)** $xJ'_n(x) = -nJ_n(x) + xJ_{n-1}(x).$

**[Jodhpur 82, 83; Meerut 80; Agra 82, 89; Kanpur 82, 84; Rohilkhand 83]**

**Proof.** As in I, we have

$$xJ'_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{(n+2r)}{r!.\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(2n+2r-n)}{r!.\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= -n \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!.\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$+ \sum_{r=0}^{\infty} (-1)^r \frac{(2n+2r)}{r!.\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= -nJ_n(x) + \sum_{r=0}^{\infty} (-1)^r \frac{2\,(n+r)}{r!\,(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} . \left(\frac{x}{2}\right)$$

$$= -nJ_n(x) + x \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= -nJ_n(x) + x \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(\overline{n-1}+r+1)} \left(\frac{x}{2}\right)^{\overline{n-1}+2r}$$

$$= -nJ_n(x) + xJ_{n-1}(x).$$

Hence $xJ'_n(x) = -nJ_n(x) + xJ_{n-1}(x)$

which may also be written as

$$\frac{d}{dx}(x^n J_n) = x^n J_{n-1}.$$

**(III)** $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x).$

**[Rohilkhand 82, 83; Poona 70; Meerut 84, 86, 86(R), 88; M.Sc. phy. 80 (S); Kanpur 81, 83; G.N.U.A. 81; Raj. 79, 81; Jodhpur 81, 84]**

**Proof.** Recurrence formula I and II, are

$$xJ'_n(x) = nJ_n(x) - xJ_{n+1}(x)$$

and

$$xJ'_n(x) = -nJ_n(x) + xJ_{n-1}(x).$$

Adding we have

$$2xJ'_n(x)=x\,[J_{n-1}(x)-J_{n+1}(x)].$$

Hence $\mathbf{2J'_n(x)=J_{n-1}(x)-J_{n+1}(x)}$.

**Aliter.** $J_n(x)=\sum_{r=0}^{\infty}(-1)^r\dfrac{1}{r\,!\,\Gamma(n+r+1)}\left(\dfrac{x}{2}\right)^{n+2r}$

$$2J'_n(x)=\sum_{r=0}^{\infty}(-1)^r\frac{2}{r\,!\,\Gamma(n+r+1)}\,(n+2r)\frac{1}{2}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=\sum_{r=0}^{\infty}(-1)^r\frac{(n+r+r)}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=\sum_{r=0}^{\infty}(-1)^r\frac{(n+r)}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$+\sum_{r=0}^{\infty}(-1)^r\frac{r}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r\,!\,\Gamma(n+r)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$-\sum_{r=0}^{\infty}(-1)^{r-1}\frac{1}{(r-1)\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r\,!\,\Gamma(\overline{n-1}+r+1)}\left(\frac{x}{2}\right)^{\overline{n-1}+2r}$$

$$-\sum_{s=0}^{\infty}(-1)^s\frac{1}{s\,!\,\Gamma(\overline{1+n}+s+1)}\left(\frac{x}{2}\right)^{n+1+2s}$$

Putting $r-1=s$

$=J_{n-1}(x)-J_{n+1}(x)$

Hence $\mathbf{2J'_n(x)=J_{n-1}(x)-J_{n+1}(x)}$.

**(IV) $2nJ_n(x)=x\,[J_{n-1}(x)+J_{n+1}(x)]$**

**[G.N.U.A. 80; Agra 84; Kanpur 81, 82, 86; Raj. 81; Meerut 80 (S), 82, 84 (P), 88; Jodhpur 84]**

**Proof.** Writting Recurrence formula I and II, we have

$$xJ'_n(x)=nJ_n(x)-xJ_{n+1}(x)$$

and $$xJ'_n(x)=-nJ_n(x)+xJ_{n-1}(x).$$

Subtracting, we have

$$0=2nJ_n(x)-x\,[J_{n+1}(x)+J_{n-1}(x)].$$

Hence $\mathbf{2nJ_n(x)=x\,[J_{n+1}(x)+J_{n-1}(x)]}$.

**Aliter.** $J_n(x)=\sum_{r=0}^{\infty}(-1)^r\dfrac{1}{r\,!\,\Gamma(n+r+1)}\left(\dfrac{x}{2}\right)^{n+2r}$

$$\therefore \quad 2nJ_n(x)=\sum_{r=0}^{\infty}(-1)^r\frac{(2n+2r-2r)}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$=\sum_{r=0}^{\infty}(-1)^r\frac{2\,(n+r)}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$-\sum_{r=0}^{\infty}(-1)^r\frac{2r}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$=x\sum_{r=0}^{\infty}(-1)^r\frac{1}{r\,!\,\Gamma(\overline{n-1}+r-1)}\left(\frac{x}{2}\right)^{\overline{n-1}+2r}$$

$$+x\sum_{r=0}^{\infty}(-1)^{r-1}\frac{1}{(r-1)\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=xJ_{n-1}(x)+x\sum_{s=0}^{\infty}(-1)^s\frac{1}{s\,!\,\Gamma(n+1+s+1)}$$

$$\times\left(\frac{x}{2}\right)^{\overline{n+1}+2s} \qquad \text{Putting } r-1=s$$

$$=xJ_{n-1}(x)+xJ_{n+1}(x)$$

**Hence $2nJ_n(x)=x\,[J_{n-1}(x)+J_{n+1}(x)]$.**

**(V) $\dfrac{d}{dx}[x^{-n}J_n(x)]=x^{-n}J_{n+1}(x)$.**

**[Jodhpur 84, 85; Kanpur 71; Meerut 82, (P); I.A.S. 80]**

**Proof.** $\dfrac{d}{dx}[x^{-n}J_n(x)]=-nx^{-n-1}J_n(x)+x^{-n}J_n'(x)$

$$=x^{-n-1}[-nJ_n(x)+xJ'_n(x)]$$

$$=x^{-n-1}[\quad nJ_n(x)+\{nJ_n(x)-xJ_{n+1}(x)\}]$$

From Rec. for I

$$=x^{-n-1}[-xJ_{n+1}(x)]$$

$$=x^{-n}.J_{n+1}(x).$$

**Hence $d/dx\,[x^{-n}J_n(x)]=-x^{-n}J_{n+1}(x)$.**

**(VI) $d/dx\,[x^nJ_n(x)]=x^nJ_{n-1}(x)$.**

**[Agra 78; Meerut 77, 83 (P); Raj. 85]**

**Proof.** $\dfrac{d}{dx}[x^nJ_n(x)]=nx^{n-1}J_n(x)+x^nJ'_n(x)$

$$=x^{n-1}[n\,J_n(x)+J'_n(x)]$$

$$=x^{n-1}[nJ_n(x)+\{-nJ_n(x)+xJ_{n-1}(x)\}]$$

$$=x^{n-1}[xJ_{n-1}(x)] \qquad \text{From Rec. for II}$$

$$=x^nJ_{n-1}(x).$$

**Hence $\quad d/dx\,[x^nJ_n(x)]=x^nJ_{n-1}(x)$.**

## § 5·7. Generating function for $J_n(x)$.

*Prove that when n is a Positive integer $J_n(x)$ is the coefficient*

*of $z^n$ in the expansion of* $e^{x\left(z-\frac{1}{z}\right)/2}$ *in ascending and decending powers of $z$. Also prove that $J_n(x)$ is the coefficient of $z^{-n}$ multiplied by $(-1)^n$ in the expansion of above experssion.*

**[G.N.U.A. 80; Agra 78, 81, 82; Kanpur 71, 82, 84, 86, 87; Meerut 78, 81 (P), 83, 86 (R), 88, 89; Jodhpur 82, 84, 86; Raj. 82, 84; Rohilkhand 82, 84; Gorakhpur 84]**

**Proof.** $e^{x\left(z-\frac{1}{z}\right)/2}=e^{xz/2}.e^{-x/2z}$

$$=\left[1+\frac{xz}{2}+\frac{1}{2!}\left(\frac{xz}{2}\right)^2+\ldots+\frac{1}{n!}\left(\frac{xz}{2}\right)^n+\frac{1}{(n+1)!}\left(\frac{xz}{2}\right)^{n+1}+\ldots+\frac{1}{(n+2)!}\left(\frac{xz}{2}\right)^{n+2}+\ldots\right]\left[1-\frac{x}{2z}+\frac{1}{2!}\left(\frac{x}{2z}\right)^2+\ldots+\frac{(-1)^n}{n!}\left(\frac{x}{2z}\right)^n+\frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2z}\right)^{n+1}+\frac{(-1)^{n+2}}{(n+2)!}\left(\frac{x}{2z}\right)^{n+2}+\ldots\right]$$

Coefficient of $z^n$ in this product

$$=\frac{1}{n!}\left(\frac{x}{2}\right)^n-\frac{1}{(n+1)!}\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+1}+\frac{1}{(n+2)!}\frac{1}{2!}\cdot\left(\frac{x}{2}\right)^2\left(\frac{x}{2}\right)^{n+2}+\ldots$$

$$=\frac{1}{n!}\left(\frac{x}{2}\right)^n-\frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2}+\frac{1}{2!\,(n+2)!}\cdot\left(\frac{x}{2}\right)^{n+4}+\ldots$$

$$=\frac{(-1)^0}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n+\frac{(-1)}{1!\,\Gamma(n+2)}\cdot\left(\frac{x}{2}\right)^{n+2}+\frac{(-1)^2}{2!\cdot\Gamma(n+3)}\cdot\left(\frac{x}{2}\right)^{n+4}+\ldots$$

$$=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\cdot\Gamma(n+r+1)}\cdot\left(\frac{x}{2}\right)^{n+2r}$$

$=J_n(x)$.

Similarly, the coefficient of $z^{-n}$ in the product

$$=\frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n+\frac{(-1)^{n+1}}{(n+1)!}\frac{x}{2}\cdot\left(\frac{x}{2}\right)^{n+1}+\frac{(-1)^{n+2}}{(n+2)!}\frac{1}{2!}\left(\frac{x}{2}\right)^2\left(\frac{x}{2}\right)^{n+2}+\ldots$$

$$=(-1)^n\left[\frac{1}{n!}\left(\frac{x}{2}\right)^n+\frac{(-1)}{\Gamma(n+2)}\left(\frac{x}{2}\right)^{n+2}\right.$$
$$\left.+\frac{(-1)^2}{2!\,\Gamma(n+3)}\left(\frac{x}{2}\right)^{n+4}+\ldots\right]$$
$$=(-1)^n J_n(x).$$

**Note.** In the above product the term independent of $z$ is

$$1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots=J_0(x).$$

Hence $e^{x(z-1/z)/2}=J_0+\left(z-\frac{1}{z}\right)J_1+\left(z^2+\frac{1}{z^2}\right)J_2+\ldots$

$$+\left[z^n+(-1)^n\frac{1}{z^n}\right]J_n+\ldots$$

$$=\sum_{-\infty}^{\infty} z^n J_n(x).$$ [**Kanpur 86, 87 ; Meerut 78, 83**]

since $J_{-n}(x)=(-1)^n J_n(x)$. See Ex. 1 below.

## EXAMPLES

**Ex. 1.** *Show that when n is a positive integer*

(*i*) $J_{-n}(x)=(-1)^n J_n(x)$

[**Rohilkhand 86 ; Meerut 81, 85, 86 (R)**]

*and* (*ii*) $J_n(-x)=(-1)^n J_n(x)$ *for +ive or −ive integers.*

**Proof.** (i) We have

$$J_{-n}(x)=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{-n+2r}\cdot\frac{1}{r!\,\Gamma(-n+r+1)}.$$

Since if $p$ is an integer, then $\Gamma(-p)$ is infinity for $p>0$, ∴ we get terms in $J_{-n}$ equal to zero till $-n+r+1<1$ *i.e.* $r<n$.

Hence we can write

$$J_{-n}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

$$=\sum_{s=0}^{\infty}\frac{(-1)^{n+s}}{(n+s)!\,\Gamma(s+1)}\left(\frac{x}{2}\right)^{n+2r}$$

Putting $r=n+s$.

$$=(-1)^n\sum_{s=0}^{\infty}\frac{(-1)^s}{\Gamma(n+s+1)\cdot s!}\left(\frac{x}{2}\right)^{n+2s}$$

$$=(-1)^n J_n(x).$$

(ii) We know that $J_n(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

**Case I. Let n be a +ive integer.** Replacing $x$ by $-x$, we have

$$J_n(-x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r\,!\,\Gamma(n+r+1)}\left(\frac{-x}{2}\right)^{n+2r}$$

$$=(-1)^n\sum_{r=0}^{\infty}(-1)^r\frac{1}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$=(-1)^n J_n(x). \qquad [\text{Since } (-1)^{2r}=1]$$

**Case II. If n is −ive integer, say** $n=-m$ where $m$ is a +ive integer.

$$\therefore\quad J_n(x)=J_{-m}(x)$$

$$=(-1)^m J_m(x) \qquad [\text{as in (i) Part}]$$

Replacing $x$ by $-x$

$$J_n(-x)=(-1)^m J_m(-x)$$

$$=(-1)^m(-1)^m J_m(x) \qquad [\text{as in case I}]$$

$$=(-1)^m J_{-m}(x)$$

$$=(-1)^{2m}\cdot(-1)^{-m}\cdot J_{-m}(x)$$

$$=(-1)^{-m} J_{-m}(x) \qquad [\because\ (-1)^{2m}=1]$$

$$=(-1)^n J_n(x).$$

Hence $J_n(-x)=(-1)^n J_n(x)$ for +ive or −ive integers.

**Proved.**

**Ex. 2.** *Prove that*

(*i*) $J_0'=-J_1$. **[Meerut M. Sc. Phy. 80 (S)]**

(*ii*) $J_2=J_0''-x^{-1}J_0'$

*and* (*iii*) $J_2-J_0=2J_0''$.

**Proof.** (i) From Recurrence formula *I*, we have

$$xJ'_n=nJ_n-xJ_{n+1}.$$

Putting $n=0$, we have

$$xJ'_0=-xJ_1.$$

$$\therefore\quad J_0'=-J_1.$$

(ii) From Recurrence formula I.

$$xJ'_n=nJ_n-xJ_{n+1}.$$

Putting $n=1$, we have

$$xJ'_1=J_1-xJ_2. \qquad \ldots(1)$$

But from (i) $J'_0=-J_1$.

$\therefore$ Differentiating, we have

$$J_0''=-J'_1.$$

Substituting these results in (1), we have

$$-xJ''_0=-J'_0-xJ_2$$

or
$$xJ_2 = xJ''_0 - J'_0,$$
$$\therefore \quad \mathbf{J_2 = J''_0 - x^{-1} J'_0}$$

(iii) From Recurrence formula III, we have
$$2J'_n = J_{n-1} - J_{n+1}. \quad \text{...(A)}$$

Differentiating both sides w.r.t. '$x$' and multiplying by 2, we have
$$2^2 J''_n = 2J'_{n-1} - 2J'_{n+1}. \quad \text{...(B)}$$

From (A), replacing $n$ by $(n-1)$ and $(n+1)$ respectively, we have
$$2J'_{n-1} = J_{n-2} - J_n$$
and
$$2J'_{n+1} = J_n - J_{n+2}.$$

Substituting these values in (B), we have
$$2^2 J''_n = (J_{n-2} - J_n) - (J_n - J_{r+2})$$
$$= J_{n-2} - 2J_n + J_{n+2}$$

Putting $n=0$
$$2^2 J''_0 = J_{-2} - 2J_0 + J_2$$
$$= (-1)^2 J_2 - 2J_0 + J_2 \text{ since } J_{-n} = (-1)^n J_n$$
$$= 2J_2 - 2J_0.$$

Hence $\quad \mathbf{2J''_0 = J_2 - J_0}.$

**Ex. 3.** *Show that*

(i) $J_{-1/2}(x) = \sqrt{\left(\dfrac{2}{\pi x}\right)} . \cos x,$

**[Kanpur 81 ; Meerut 83 ; Jiwaji 82 ; Poona 89]**

(*ii*) $J_{1/2}(x) = \sqrt{\left(\dfrac{2}{\pi x}\right)} \cdot \sin x,$

**[Agra 81, 86; Meerut 78 ; Kanpur 81, 83, 86, 87 ; Jiwaji 82 ; Rohilkhand 80]**

*and* (*iii*) $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \dfrac{2}{\pi x}.$ **[Meerut 90]**

**Proof.** (i) We know that
$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} + \dots\right]$$

Putting $n = -\frac{1}{2}$
$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2} \Gamma(\frac{1}{2})} \left[1 - \frac{x^2}{2(-1+2)} + \frac{x^4}{2.4.(-1+2)(-1+4)} - \dots\right]$$

$$=\sqrt{\left(\frac{2}{\pi x}\right)}\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}\cdots\right]=\sqrt{\left(\frac{2}{\pi x}\right)}\cdot\cos x.$$

Putting $n=\frac{1}{2}$, in $J_n(x)$, we have

$$J_{1/2}(x)=\frac{x^{1/2}}{2^{1/2}\,\Gamma(3/2)}\cdot\left[1-\frac{x^2}{2.(1+2)}+\frac{x^4}{2\cdot 4.(1+2)(1+4)}\cdots\right]$$

$$=\sqrt{\left(\frac{x}{2}\right)}\frac{1}{\frac{1}{2}\sqrt{\pi}}\cdot\frac{1}{x}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right]$$

$$=\sqrt{\left(\frac{2}{\pi x}\right)}\cdot\sin x.$$

(iii) Square and add (i) and (ii).

**Ex. 4.** *Establish the differential formula*

$$x^2J''_n(x)=(n^2-n-x^2)J_n(x)+xJ_{n+1}(x)\ (n=0, 1\ 2 \ldots).$$

**Proof.** Writing recurrence formula I, we have

$$xJ'_n(x)=nJ_n(x)-xJ_{n+1}(x). \qquad \ldots(1)$$

Differentiating both sides w r.t. '$x$', we have

$$xJ''_n(x)+J'_n(x)=nJ'_n(x)-xJ'_{n+1}(x)+J_{n+1}(x)$$

or $$x^2J''_n(x)=(n-1)\,xJ'_n(x)-x.xJ'_{n+1}(x)-xJ_{n+1}(x) \qquad \ldots(2)$$

From recurrence formula II.

$$xJ'_n(x)=-nJ_n(x)+xJ_{n-1}(x).$$

Writing $(n+1)$ for $n$ ; we have

$$xJ'_{n+1}(x)=-(n+1)J_{n+1}(x)+xJ_n(x) \qquad \ldots(3)$$

Substituting for $xJ'_n(x)$ from (1) and for $xJ_{n+1}(x)$ from (3) in (2), we have

$$x^2J_n''(x)=(n-1)[nJ_n(x)-xJ_{n+1}(x)$$
$$-x[-(n+1)J_{n+1}(x)+xJ_n(x)]-xJ_{n+1}(x)$$
$$=(n^2-n-x^2)J_n(x)+xJ_{n+1}(x)$$

**Ex. 5** *Prove that*

$$J_{n-1}=\frac{2}{x}[nJ_n-(n+2)J_{n+2}+(n+4)J_{n+4}\ldots]$$

*and hence deduce that* [Rohilkhand 83]

$\frac{1}{2}xJ_n=(n+1)J_{n+1}-(n+3)J_{n+3}+(n+5)J_{n+5}+\ldots$ [Agra 82]

**Proof.** From recurrence formula IV, we have

$$x(J_{n-1}+J_{n+1})=2nJ_n$$

$$\therefore\quad J_{n-1}+J_{n+1}=\frac{2}{x}\,nJ_n. \qquad \ldots(1)$$

Replacing $n$ by $(n+2)$, and changing sign, we have

$$-J_{n+1}-J_{n+3}=\frac{-2}{x}(n+2)J_{n+2}. \qquad \ldots(2)$$

Again replacing $n$ by $(n+4)$ in (1), we have

$$J_{n+3}+J_{n+5}=\frac{2}{x}(n+4)\ J_{n+4}. \qquad \text{...(3)}$$

Again replacing $n$ by $(n+6)$ in (1) and changing sign, we have

$$-J_{n+5}-J_{n+7}=\frac{2}{x}(n+6)\ J_{n+6}. \qquad \text{...(4)}$$

... ... ... ...

... ... ... ...

Adding (1), (2), (3), (4) etc., we have

$$J_{n-1}=\frac{2}{x}[nJ_n-(n+2)\ J_{n+2}+(n+4)\ J_{n+4}-(n+6)\ J_{n+6}\dots]$$

Since $J_n \to 0$ as $n \to \infty$.

**Deduction.** Replacing $n$ by $(n+1)$, we have

$$\frac{x}{2}\ J_n=(n+1)\ J_{n+1}-(n+3)\ J_{n+3}+(n+5)J_{n+5}\dots$$

**Ex. 6.** *Prove that*

$$J_n'=\frac{2}{x}\left[\frac{n}{2}\ J_n-(n+2)\ J_{n+2}+(n+4)\ J_{n+4}\dots\right] \quad \text{[Rohilkhand 85]}$$

**Proof.** From recurrence formula II, we have

$$J_n'=-\frac{n}{x}\ J_n+J_{n-1}.$$

Substituting the value of $J_{n-1}$ from Ex. 5.

$$J_n'=-\frac{n}{x}\ J_n+\frac{2}{x}[nJ_n-(n+2)\ J_{n+2}+(n+4)\ J_{n+4}\dots]$$

$$=\frac{2}{x}\left[\frac{n}{2}\ J_n-(n+2)\ J_{n+2}+(n+4)\ J_{n+4}\dots\right]$$

**Ex. 7.** *Prove that*

$$\frac{d}{dx}[J_n^2+J^2_{n+1}]=2\left(\frac{n}{x}\ J_n^2-\frac{n+1}{x}\ J^2_{n+1}\right)$$

**[Agra 80, 83, 85; Meerut 79; Kanpur 83, 85; Raj 80, 86]**

or $$J_nJ_n'+J_{n+1}\ J'_{n+1}=\frac{1}{x}[nJ_n^2-(n+1)\ J^2_{n+1}] \quad \text{[Kanpur 84, 85]}$$

**Proof.** $$\frac{d}{dx}(J_n^2+J^2_{n+1})=2J_nJ_n'+2J_{n+1}J'_{n+1} \qquad \text{...(1)}$$

From recurrence formula I, we have

$$J_n'=\frac{n}{x}\ J_n-J_{n+1}. \qquad \text{...(2)}$$

Also from recurrence formula II, we have

$$J_n'=-\frac{n}{x}\ J_n+J_{n-1}.$$

Replacing $n$ by $(n+1)$, we have

$$J'_{n+1}=-\frac{n+1}{x}J_{n+1}+J_n. \qquad ...(3)$$

Substituting the values of $J_n'$ and $J'_{n+1}$ from (2) and (3) in (1), we have

$$\frac{d}{dx}(J_n^2+J^2_{n+1})=2J_n\left(\frac{n}{x}J_n-J_{n+1}\right)+2J_{n+1}\left(-\frac{n+1}{x}J_{n+1}+J_n\right)$$

$$=2\left(\frac{n}{x}J_n^2-\frac{n+1}{x}J^2_{n+1}\right)$$

**Ex.8.** *Prove that*

$$J_0^2+2(J_1^2+J_2^2+J_3^2+\ldots)=1.$$

**[Agra 83, 85 ; Kanpur 83, 84, 85]**

*Deduce that*

$$|J_0(x)|\leqslant 1,\ |J_n(x)|\leqslant {}^{-1/2},\ (n\geqslant 1)$$

**Proof.** In the last example 7, we have proved that

$$\frac{d}{dx}(J_n^2+J^2_{n+1})=2\left(\frac{n}{x}J_n^2-\frac{n+1}{x}J^2_{n+1}\right)$$

Putting $n=0, 1, 2, 3\ldots$etc., we have

$$\frac{d}{dx}(J_0^2+J_1^2)=2\left(0-\frac{1}{x}J_1^2\right)$$

$$\frac{d}{dx}(J_1^2+J_2^2)=2\left(\frac{1}{x}J_1^2-\frac{2}{x}J_2^2\right)$$

$$\frac{d}{dx}(J_2^2+J_3^2)=2\left(\frac{2}{x}J_2^2-\frac{3}{x}J_3^2\right)$$

... ... ... ... ... ...

... ... ... ... ... ... etc.

Adding, we have

$$\frac{d}{dx}[J_0^2+2(J_1^2+J_2^2+J_3^2+\ \ )]=0.$$

Since $\qquad J_n\to\infty$

as $\qquad n\to\infty.$

Integrating, we have

$$J_0^2+2(J_1^2+J_2^2+J_3^2+\ldots)=c \text{ (constant)},$$

Putting $\quad x=0.$

$$(J_0^2)_{x=0}=c \qquad \text{since } (J_1)_{x=0}=0$$

or $\qquad 1=c \qquad (J_2)_{x=0}=0$ etc.

Hence $\qquad J_0^2+2(J_1^2+J_2^2+J_3^2+\ldots)=1.$

**Deduction.** We have proved that

$$J_0^2+2(J_1^2+J_2^2+J_3^2\ldots+J_n^2+\ldots)=1.$$

Since $J_1^2, J_2^2, J_3^2$...are all positive or zero

$\therefore \quad J_0^2 \leqslant 1 \qquad$ hence $|J_0| \leqslant 1$

i.e. $\qquad |J_0(x)| \leqslant 1.$

Also $\qquad |2J_n^2| \leqslant 1$

or $\qquad |J_n| \leqslant \dfrac{1}{\sqrt{2}}$

Hence $|J_n| \leqslant 2^{-1/2}$, *i.e.* $|J_n(x)| \leqslant 2^{-1/2}$ (for $n \geqslant 1$).

**Ex. 9.** *Prove that*

$$\frac{d}{dx}(xJ_nJ_{n+1})=x(J_n^2-J^2_{n+1}).$$

**[Meerut 82 (P), 84, 87; Raj. 81 ; Kanpur 80, 87 ; Agra 81 88] Gorakhpur 84]**

**Proof.** $\dfrac{d}{dx}(xJ_n J_{n+1})$

$$=J_n J_{n+1}+x(J_n' J_{n+1}+J_n J'_{n+1})$$
$$=J_n J_{n+1}+(xJ_n') J_{n+1}+J_n(xJ'_{n+1}) \qquad ...(1)$$

From Recurrence formula I and II, we have

$$xJ'_n=nJ_n-xJ_{n+1}. \qquad ...(2)$$

and
$$xJ_n'=-nJ_n+xJ_{n-1}. \qquad ...(3)$$

Replacing $n$ by $(n+1)$ in (3), we have

$$xJ'_{n+1}=-(n+1)J_{n+1}+xJ_n \qquad ...(4)$$

Substituting the values of $xJ'_n$ and $xJ'_{n+1}$ from (2) and (4), in (1), we have

$$\frac{d}{dx}(xJ_nJ_{n+1})=J_nJ_{n+1}+(nJ_n-xJ_{n+1})J_{n+1}$$
$$+J_n\{-(n+1)J_{n+1}+xJ_n\}$$
$$=x(J_n^2-J^2_{n+1})$$

**Ex. 10.** *Prove that*

$$x=2J_0J_1+6J_1J_2+\ldots+2(2n+1)J_nJ_{n+1}+\ldots \qquad \textbf{[Agra 81]}$$

**Proof.** In example 9, we have already proved that

$$\frac{d}{dx}(xJ_nJ_{n+1})=x(J_n^2-J^2_{n+1})$$

Putiting $n=0, 1, 2, 3$...we have

$$\frac{d}{dx}(xJ_0J_1)=x(J_0^2-J_1^2) \qquad ...(1)$$

$$\frac{d}{dx}(xJ_1J_2)=x(J_1^2-J_2^2) \qquad ...(2)$$

$$\frac{d}{dx}(xJ_2J_3)=x(J_2^2-J_3^2) \qquad ...(3)$$

$$\frac{d}{dx}(xJ_3J_4)=x(J_3^2-J_4^2) \qquad \text{...(4)}$$

... ... ... ...

... ... ... ...

Multiplying (1), (2), (3),...by 1, 3, 5,...respectively and adding we have

$$\frac{d}{dx}[x(J_0J_1+3J_1J_2+5J_2J_3+\ldots)]$$

$$=x[(J_0^2-J_1^2)+3(J_1^2-J_2^2)+5(J_2^2-J_3^2)+\ldots]$$

$$=x[J_0^2+2(J_1^2+J_2^2+\ldots)]$$

$$=x.\,1=x. \qquad \text{(from Ex. 1)}$$

Integrating both sides, we have

$$x(J_0J_1+3J_1J_2+5J_2J_3+\quad)=\frac{x^2}{2}+c \text{ (constant)}.$$

Putting $x=0$, $0=0+c$. $\therefore$ $c=0$.

Hence $x(J_0J_1+3J_1J_2+5J_2J_3+\ldots)=\dfrac{x^2}{2}$

or $\quad 2J_0J_1+5J_1J_2+10J_2J_3+\ldots+2.(2n+1)\,J_nJ_{n+1}+\ldots=x.$

**Ex. 11.** *From the Recurrence formula*

$$2J_n'=J_{n-1}-J_{n+1}$$

*deduce the result*

$$2^r\,J_n^r=J_{n-r}-rJ_{n-r+2}+\frac{r(r-1)}{2!}J_{n-r+4}-\ldots+(-1)^r\,J_{n+r}.$$

**Proof.** Recurrence formula is

$$2J_n'=J_{n-1}-J_{n+1} \qquad \text{...(1)}$$

Differentiating w.r.t. '$x$', we have

$$2J_n''=J'_{n-1}-J'_{n+1}$$

$$\therefore\quad 2^2J''_n=2J'_{n-1}-2J'_{n+1} \qquad \text{...(1)}$$

Substituting the values of $2J'_{n-1}$ and $2J'_{n+1}$ from (1) obtained by replacing $n$ by $(n-1)$ and $(n+1)$ respectively

$$=(J_{n-2}-J_n)-(J_n-J_{n+2})$$

$$=J_{n-2}-2J_n+J_{n+2}.$$

Differentiating (2) again, and multiplying by 2, we have

$$2^3J_n'''=2J'_{n-2}-2^2J'_n+2J'_{n+2}.$$

[Substituting the values of $2J'_n$, $2J'_{n-2}$, $2J'_{n+2}$ from (1), the last two obtained by replacing $n$ by $(n-2)$ and $(n+2)$ respectively in (1)

$$=(J_{n-3}-J_{n-1})-2(J_{n-1}-J_{n+1})+(J_{n+1}-J_{n+3})$$

$$=J_{n-3}-3J_{n-1}+3J_{n+1}-J_{n+3}$$

$$=J_{n-3}-{}^3C_1J_{n-1}+{}^3C_2J_{n+1}-{}^3C_3J_{n+3}.$$

Applying the same process again, and again, we have

$$2^rJ_n^r=J_{n-r}-{}^rC_1J_{n-r+2}+{}^rC_2J_{n-r+4}+\ldots+(-1)^r.{}^rC_rJ_{n+r}$$

$$=J_{n-r}-r\,J_{n-r+2}+\frac{r(r-1)}{2\,!}\,J_{n-r+4}+\ldots+(-1)^r\,J_{n+r}.$$

**Ex. 12.** *Prove that*

$$\sqrt{\left(\frac{\pi x}{2}\right)}\,J_{3/2}\,(x)=\frac{1}{x}\sin x-\cos x.$$

**[Meerut 79, 81(P); Raj. 79, 81, 82, 85; Rohilkhand 80, 89]**

**Proof.** We know that

$$J_n\,(x)=\frac{x^n}{2^n\Gamma(n+1)}\left[1-\frac{x^2}{2\,(2n+2)}+\frac{x^4}{2.4.(2n+2)\,(2n+4)}\ldots\right]$$

Putting $n=3/2$, we have

$$J_{3/2}(x)=\frac{x^{3/2}}{2^{3/2}\,\Gamma(5/2)}\left[1-\frac{x^2}{2.5}+\frac{x^4}{2.4.5.7}-\frac{x^6}{2.4.5.6.7.9}\ldots\right]$$

$$=\frac{x\sqrt{x}}{2\sqrt{2}.3/2.1/2\sqrt{\pi}}\left[1-\frac{x^2}{2.5}+\frac{x^4}{2.4.5.7}\right.$$

$$\left.-\frac{x^6}{2.4.5.6.7.9}+\ldots\right]$$

$$\therefore\quad\sqrt{\left(\frac{\pi x}{2}\right)}\,J_{3/2}(x)=\frac{1}{3}\left[x^2-\frac{x^4}{2.5}+\frac{x^6}{2.4.5.7}-\frac{x^8}{2.4.5.6.7.9}+\ldots\right]$$

$$=\frac{2x^2}{3\,!}-\frac{4x^4}{5\,!}+\frac{6x^6}{7\,!}-\frac{8x^8}{9\,!}+\ldots$$

$$=\left(\frac{1}{2\,!}-\frac{1}{3\,!}\right)x^2-\left(\frac{1}{4\,!}-\frac{1}{5\,!}\right)x^4$$

$$+\left(\frac{1}{6\,!}-\frac{1}{7\,!}\right)x^6-\left(\frac{1}{8\,!}-\frac{1}{9\,!}\right)x^8\ldots$$

$$=\left(\frac{x^2}{2\,!}-\frac{x^4}{4\,!}+\frac{x^6}{6\,!}\ldots\right)+\left(-\frac{x^2}{3\,!}+\frac{x^4}{5\,!}-\frac{x^6}{7\,!}\ldots\right)$$

$$=-\left(1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}\ldots\right)$$

$$+\frac{1}{x}\left[x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\frac{x^7}{7\,!}\ldots\right]$$

$$=-\cos x+\frac{1}{x}\sin x.$$

Hence $\sqrt{\left(\frac{\pi x}{2}\right)}\,J_{3/2}\,(x)=\frac{1}{x}\sin x-\cos x.$

**Ex. 13.** *Prove that*

$$\frac{d}{dx}\left(\frac{J_{-n}}{J_n}\right)=-\frac{2\sin n\pi}{\pi x J_n^2}.$$

[Agra 78, 82; Raj. 77, 83; Kanpur 87; Jodhpur 83]

**Proof.** Since $J_n$ and $J_{-n}$ are the solutions of the Bessel's differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0.$$

$$\therefore \quad \frac{d^2J_n}{dx^2}+\frac{1}{x}\frac{dJ_n}{dx}+\left(1-\frac{n^2}{x^2}\right)J_n=0. \quad \text{...(i)}$$

and

$$\frac{d^2J_{-n}}{dx^2}+\frac{1}{x}\frac{dJ_{-n}}{dx}+\left(1-\frac{n^2}{x^2}\right)J_{-n}=0. \quad \text{...(ii)}$$

Multiplying (i) by $J_{-n}$ and (ii) by $J_n$ and subtracting, we have

$$\left(J_{-n}\frac{d^2J_n}{dx^2}-J_n\frac{d^2J_{-n}}{dx^2}\right)+\frac{1}{x}\left(J_{-n}\frac{dJ_n}{dx}-J_n\frac{dJ_{-n}}{dx}\right)=0.$$

$$\therefore \quad \frac{J_{-n}\dfrac{d^2J_n}{dx^2}-J_n\dfrac{d^2J_{-n}}{dx^2}}{J_{-n}\dfrac{dJ_n}{dx}-J_n\dfrac{dJ_{-n}}{dx}}=-\frac{1}{x}.$$

Integrating both the sides w.r.t. '$x$', we have

$$\log\left(J_{-n}\frac{dJ_n}{dx}-J_n\frac{dJ_{-n}}{dx}\right)=-\log x+\log A$$

where $A$ is an arbitrary constant.

$$\therefore \quad J_{-n}\frac{dJ_n}{dx}-J_n\frac{dJ_{-n}}{dx}=\frac{A}{x}. \quad \text{...(iii)}$$

$$\therefore \quad \frac{1}{2^{-n}\Gamma(-n+1)}\left[x^{-n}-\frac{x^{-n+2}}{2.(-2n+2)}+\frac{x^{-n+4}}{2.4(-2n+2)(-2n+4)}\cdots\right]$$

$$\times\frac{1}{2^n\Gamma(n+1)}\left[nx^{n-1}-\frac{(n+2)\,x^{n+1}}{2.(2n+2)}+\frac{(n+4)\,x^{n+3}}{2.4(2n+2)(2n+4)}\cdots\right]$$

$$-\frac{1}{2^n\Gamma(n+1)}.\left[x^n-\frac{x^{n+2}}{2.(2n+2)}+\frac{x^{n+4}}{2.4(2n+2)(2n+4)}\right]$$

$$\times\frac{1}{2^{-n}\Gamma(-n+1)}.\left[-nx^{-n-1}-\frac{(-n+2)\,x^{-n+1}}{2.(-2n+2)}+\frac{(-n+4).x^{-n+3}}{2.4(-2n+2)(-2n+4)}\right]$$

$$=\frac{A}{x}$$

Comparing coefficient of $\frac{1}{x}$ on both the sides, we have

$$\frac{1}{\Gamma(-n+1)\,\Gamma(n+1)}\,[n-(-n)]=A.$$

$$\therefore \quad A = \frac{2n}{\Gamma(-n+1)\,\Gamma(n+1)}$$

$$= \frac{2}{\Gamma(1-n)\,\Gamma(n)}$$

$$= \frac{2}{\dfrac{\pi}{\sin n\pi}} \qquad \text{Since } \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$= \frac{2\sin n\pi}{\pi}.$$

Hence from (iii), we have

$$J_{-n}\frac{dJ_n}{dx} - J_n\frac{dJ_{-n}}{dx} = \frac{2\sin n\pi}{\pi x}.$$

Dividing both sides by $-J^2_n$

$$\frac{\dfrac{dJ_{-n}}{dx}.J_n - J_{-n}.\dfrac{dJ_n}{dx}}{J^2_n} = -\frac{2\sin n\pi}{\pi x J^2_n}.$$

$$\frac{d}{dx}\left(\frac{J_{-n}}{J_n}\right) = -\frac{2\sin n\pi}{\pi x J^2_n}.$$

**Ex. 14.** *Prove that*

$$\frac{J_{n+1}}{J_n} = \frac{(x/2)}{(n+1)}\,\frac{(x/2)^2}{-(n+2)}\,\frac{(x/2)^2}{-(n+3)}\cdots$$

**Proof.** From recurrence formula IV, we have

$$J_{n-1} + J_{n+1} = \frac{2n}{x}J_n$$

$$\therefore \quad J_{n-1} = \frac{2n}{x}J_n - J_{n+1}.$$

Replacing $n$ by $(n+1)$, we have

$$J_n = \frac{2(n+1)}{x}J_{n+1} - J_{n+2}$$

$$\therefore \quad \frac{J_n}{J_{n+1}} = \frac{2(n+1)}{x} - \frac{J_{n+2}}{J_{n+1}} \qquad \ldots(i)$$

$$\therefore \quad \frac{J_{n+1}}{J_n} = \frac{1}{\dfrac{J_n}{J_{n+1}}} = \frac{1}{\dfrac{2(n+1)}{x} - \dfrac{J_{n+2}}{J_{n+1}}}$$

$$= \frac{1}{\dfrac{2(n+1)}{x} - \dfrac{1}{\dfrac{J_{n+1}}{J_{n+2}}}}$$

$$=\cfrac{1}{\cfrac{2(n+1)}{x}-\cfrac{1}{\cfrac{2(n+2)}{x}-\cfrac{J_{n+3}}{J_{n+2}}}}$$

(with the help of (i) replacing $n$ by $n+1$)

$$=\cfrac{1}{\cfrac{2(n+1)}{x}-\cfrac{1}{\cfrac{2(n+2)}{x}-\cfrac{1}{\cfrac{J_{n+2}}{J_{n+3}}}}}$$

$$=\cfrac{1}{\cfrac{2(n+1)}{x}-\cfrac{1}{\cfrac{2(n+2)}{x}-\cfrac{1}{\cfrac{2(n+3)}{x}}}}\cdots$$

(with the help of (i) again and again)

$$=\cfrac{x/2}{(n+1)-\cfrac{x/2}{\cfrac{2(n+2)}{x}-\cfrac{1}{\cfrac{2(n+3)}{x}}}}$$

$$=\cfrac{x/2}{(n+1)-\cfrac{(x/x)^2}{(n+2)-\cfrac{x/2}{\cfrac{2(n+3)}{x}}}}\cdots$$

$$=\frac{(x/2)}{(n+1)}\,\frac{(x/2)^2}{-(n+2)}\,\frac{(x/2)^2}{-(n+3)}\cdots$$

**Ex. 15.** *Verify directly from the representation*

$$J_0(x)=\frac{1}{\pi}\int_0^{\pi} \cos (x \sin \phi)\, d\phi$$

*that $J_0(x)$ satisfies Bessel's equation in which $n=0$.*

**Proof.** Let

$$y=J_0(x)=\frac{1}{\pi}\int_0^{\pi} \cos (x \sin \phi)\, d\phi \qquad \ldots\text{(i)}$$

$$\therefore \quad \frac{dy}{dx}=-\frac{1}{\pi}\int_0^{\pi} \sin (x \sin \phi) \sin \phi\, d\phi \qquad \ldots\text{(ii)}$$

and

$$\frac{d^2y}{dx^2}=-\frac{1}{\pi}\int_0^{\pi} \cos (x \sin \phi) \sin^2 \phi\, d\phi \qquad \ldots\text{(iii)}$$

Evaluating the R.H.S. of (ii) by the method of integration by parts, we have

$$\frac{dy}{dx}=-\frac{1}{\pi}\left[\left\{-\sin (x \sin \phi) \cos \phi\right\}_0^\pi + \int_0^\pi \cos (x \text{ stn } \phi).x \cos^2 \phi \, d\phi\right]$$

$$=-\frac{x}{\pi}\int_0^\pi \cos (x \sin \phi).\cos^2 \phi \, d\phi$$

$$=-\frac{x}{\pi}\int_0^\pi \cos (x \sin \phi).(1-\sin^2 \phi) \, d\phi$$

$$=-\frac{x}{\pi}\int_0^\pi \cos (x \sin \phi) \, d\phi+\frac{x}{\pi}\int_0^\pi \cos (x \sin \phi) \sin^2 \phi \, d\phi$$

$$=-xy-x\frac{d^2y}{dx^2} \text{ from (i) and (iii)}$$

$$\therefore \quad \frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0$$

which is Bessel's equation for $n=0$.

Hence $J_0(x)=\frac{1}{x}\int_0^\pi \cos (x \sin \phi) \, d\phi$ satisfies the Bessel's equation of the zeroeth order.

**Ex. 16.** *Show that*

(*i*) $\cos x=J_0-2J_2+2J_4 \ldots\ldots$ **[Rohilkhand 80; Kanpur 83)**

*and* (*ii*) $\sin x=2J_1-2J_3+2J_5 \ldots\ldots$ **[Rohilkhand 80; Kanpur 84]**

**Proof.** We know that

$$e^{x(z-1/z)/2}=J_0+\left(z-\frac{1}{z}\right) J_1+\left(z^2+\frac{1}{z^2}\right) J_2+\left(z^3-\frac{1}{z^3}\right) J_3+\ldots$$

Putting $z=e^{i\theta}$, we have

$$e^{x.\frac{e^{i\theta}-e^{-i\theta}}{2}}=J_0+(e^{i\theta}-e^{-i\theta}) J_1+(e^{2i\theta}+e^{-2i\theta}) J_2+(e^{3i\theta}-e^{-3i\theta})J_3+\ldots$$

or $$e^{xi \sin \theta}=J_0+(2i \sin \theta) J_1+(2 \cos 2\theta) J_2+(2i \sin 3\theta) J_3+\ldots$$

or $$\cos (x \sin \theta)+i \sin (x \sin \theta) =(J_0+2 \cos 2\theta \, J_2+2 \cos 4\theta . J_4+\ldots) +i\,(2 \sin \theta \, J_1+2 \sin 3\theta \, J_3+\ldots)$$

Equating real and imaginary parts we have

$$\cos (x \sin \theta)=J_0+2 \cos 2\theta \, J_2+2 \cos 4\theta \, J_4+\ldots$$

and $$\sin (x \sin \theta)=2 \sin \theta . J_1+2 \sin 3\theta \, J_3+2 \sin 5\theta \, J_5+\ldots$$

Putting $\theta=\frac{\pi}{2}$ we have

$$\cos x = J_0 - 2J_2 + 2J_4 + \ldots$$

and $$\sin x = 2J_1 - 2J_3 + 2J_5 - \ldots$$

**Ex. 17.** *If $n > -1$, show that*

$$\int_0^x x^{-n} J_{n+1}(x)\, dx = \frac{1}{2^n \Gamma(n+1)} - x^{-n} J_n(x).$$ [Jodhpur 83]

**Proof.** From recurrence formula V, we have

$$x^{-n} J_{n+1}(x) = -\frac{d}{dx}\{x^{-n} J_n(x)\}.$$

Integrating between the limits 0 and $x$, we have

$$\int_0^x x^{-n} J_{n+1}(x)\, dx = -\left[x^{-n} J_n(x)\right]_0^x$$

$$= -x^{-n} J_n(x) + \lim_{x \to 0} \frac{J_n(x)}{x^n}$$

(Diff. Nr. and Dr. $n$ times)

$$= -x^{-n} J_n(x) + \lim_{x \to 0} \frac{J^n{}_n(x)}{n\,!}$$

$$= -x^{-n} J_n(x) + \frac{n\,!}{2^n \Gamma(n+1) . n\,!}$$

$$= -x^{-n} J_n(x) + \frac{1}{2^n \Gamma(n+1)}.$$

**Ex. 18.** *Show that when n is integral*

(*a*) $$\pi J_n = \int_0^\pi \cos(n\theta - x \sin \theta)\, d\theta$$

[Agra 79, 84, 87; Meerut 82, 84(P), 87, 89; I.A.S. 77 Raj. 80, 83, 86; Jodhpur 82, 85, 86; Kanpur 86, 87; Rohilkhand 84]

(*b*) $$\pi J_0 = \int_0^\pi \cos(x \cos \phi)\, d\phi$$

$$= \int_0^\pi \cos(x \sin \phi)\, d\phi$$ [Rohilkhand 84; Jodhpur 82]

*and hence deduce that*

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2^r.r\,!)^2}.$$

[Kanpur 83]

**Proof.** Proceeding as in Ex. 16, we get

$$\cos(x \sin \theta) = J_0 + 2J_2 \cos 2\theta + \ldots + 2J_{2m} \cos 2m\theta + \ldots \quad \ldots(i)$$

and $$\sin(x \sin \theta) = 2 \sin \theta . J_1 + 2 \sin 3\theta . J_3 + \ldots + 2J_{2m+1} \sin(2m+1)\theta + \ldots \quad \ldots(ii)$$

(a) Multiplying both sides of (i) by $\cos 2m\theta$ and then integrating between the limits 0 to $\pi$

$$\int_0^\pi \cos(x\sin\theta)\cos 2m\theta\, d\theta = J_0\int_0^\pi \cos 2m\theta\, d\theta$$

$$+2J_2\int_0^\pi \cos 2\theta \cos 2m\theta\, d\theta+\ldots$$

$$+2J_{2m}\int_0^\pi \cos^2 2m\theta\, d\theta+\ldots$$

$$=0+0+\ldots$$

$$+J_{2m}\int_0^\pi (1+\cos 4m\theta)\, d\theta+\ldots$$

$$=\pi J_{2m}.$$

Similarly we can prove that

$$\int_0^\pi \cos(x\sin\theta)\cos(2m+1)\,\theta.\,d\theta=0$$

Again multiplying both sides of (ii) by $\sin(2m+1)\,\theta$ and then integrating between the limits 0 to $\pi$.

$$\int_0^\pi \sin(x\sin\theta)\sin(2m+1)\,\theta\, d\theta$$

$$=2J_1\int_0^\pi \sin\theta.\sin(2m+1)\,\theta\, d\theta$$

$$+2J_3\int_0^\pi \sin 3\theta\sin(2m+1)\,\theta\, d\theta+\ldots$$

$$+2J_{2m+1}\int_0^\pi \sin^2(2m+1)\,\theta\, d\theta+\ldots$$

$$=0+0+\ldots+J_{2m+1}\int_0^\pi \{1-\cos 2(2m+1)\,\theta\}\, d\theta+\ldots$$

$$=J_{2m+1}\Big(\theta\Big)_0^\pi=\pi J_{2m+1}.$$

Similarly,

$$\int_0^\pi \sin(x\sin\theta)\sin 2m\theta\, d\theta=0.$$

Therefore,

$$\int_0^\pi \cos(2m\theta-x\sin\theta)\, d\theta=\int_0^\pi \cos 2m\theta.\cos(x\sin\theta)\, d\theta$$

$$+\int_0^\pi \sin 2m\theta.\sin(x\sin\theta)\, d\theta$$

$$=\pi J_{2m}.$$

Also

$$\int_0^\pi \cos[(2m+1)\,\theta-x\sin\theta]\, d\theta=\int_0^\pi \cos(2m+1)\,\theta.\cos(x\sin\theta)\, d\theta$$

$$+\int_0^\pi \sin(2m+1)\,\theta\sin(x\sin\theta)\, d\theta$$

$$=\pi J_{2m+1}.$$

Hence for all positive integral $n$ we have

$$\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta = \pi J_n.$$

If $n$ is negative, say $n=-m$, where $m$ is +ve then

$$\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta$$

$$=\int_0^\pi \cos(-m\theta - x\sin\theta)\,d\theta$$

$$=-\int_\pi^0 \cos\{-m(\pi-\phi)-x\sin(\pi-\phi)\}\,d\phi.$$

Putting $\theta=\pi-\phi$

$$=\int_0^\pi \cos\{-m\pi+(m\phi-x\sin\phi)\}\,d\phi$$

$$=\int_0^\pi \{\cos m\pi . \cos(m\phi - x\sin\phi)$$

$$+\sin m\pi \sin(m\phi - x\sin\phi)\}\,d\phi$$

$$=(-1)^m \int_0^\pi \cos(m\phi - x\sin\phi)\,d\phi$$

$$=(-1)^m \pi J_m(x).$$

Since we have proved the result for positive integer.

$$=\pi J_{-m}(x).$$

Since $J_{-m}(x)=(-1)^m J_m(x)$.

$$=\pi J_n(x).$$

Hence for all integral values of $n$

$$\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta = \pi J_n.$$

**Proved.**

(b) Putting $\theta=\frac{\pi}{2}+\phi$ in the value of $\cos(x\sin\theta)$ from (i), we have

$$\cos(x\cos\phi)=J_0-2J_2\cos 2\phi+2J_4\cos 4\phi-\ldots$$

$$\therefore \int_0^\pi \cos(x\cos\phi)\,d\phi = J_0\int_0^\pi d\theta - 2J_2\int_0^\pi \cos 2\phi\,d\phi+\ldots$$

$$=\pi J_0$$

**Proved.**

From (i) we have

$$\cos(x\sin\phi)=J_0+2J_2\cos 2\phi+2J_4\cos 4\phi+\ldots$$

$$\therefore \int_0^\pi \cos(x\sin\phi)\,d\phi = J_0\int_0^\pi d\phi+2J_2\int_0^\pi \cos 2\phi\,d\phi+\ldots$$

$$=\pi J_0.$$

**Proved.**

**Deduction.** We have proved that

$$J_0(x)=\frac{1}{\pi}\int_0^{\pi} \cos(x\cos\phi)\,d\phi$$

$$=\frac{1}{\pi}\int_0^{\pi}\left(1-\frac{x^2\cos^2\phi}{2\,!}+\frac{x^4\cos^4\phi}{4\,!}-\frac{x^6\cos^6\phi}{6\,!}+\dots\right)d\phi. \qquad \dots(1)$$

Since $\int_0^{\pi}\cos^{2r}\phi\,d\phi=\frac{1.3.5\dots(2r-1)}{2.4.6\dots(2r)}.\pi$ from definite integrals.

$\therefore$ from (1) we have

$$J_0(x)=\frac{1}{\pi}\left[\pi-\frac{x^2}{2\,!}\cdot\frac{1}{2}\pi+\frac{x^4}{4\,!}\cdot\frac{1.3}{2.4}\pi-\frac{x^6}{6\,!}\cdot\frac{1.3.5}{2.4.6}\pi+\dots\right]$$

$$=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\dots$$

$$=1-\frac{x^2}{2^2}+\frac{x^4}{2^4.(2\,!)^2}-\frac{x^6}{2^6.(3\,!)^2}+\dots$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2^r.r\,!)^2}.$$

**Ex. 19.** *Prove that*

$$J_n(x)=\frac{(x/2)^n}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_{-1}^{+1}(1-t^2)^{n-1/2}\,e^{ixt}.dt.(n>-\tfrac{1}{2})$$

*And deduce that*

$$\int_0^1(1-t^2)^{n-1/2}\cos(xt)\,dt=\frac{2^{n-1}\,\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2}).J_n(x)}{x^n}$$

[Raj. 79]

**Proof.** Let

$$I=\int_{-1}^{+1}(1-t^2)^{n-1/2}.e^{ixt}.dt$$

$$=\int_{-1}^{+1}(1-t^2)^{n-1/2}\left(\sum_{r=0}^{\infty}\frac{(ixt)^r}{r\,!}\right)dt$$

$$=\sum_{r=0}^{\infty}\frac{(ix)^r}{r\,!}\int_{-1}^{+1}(1-t^2)^{n-1/2}.t^r\,dt$$

Now, if $r$ is odd, the integrand in $I$ is an odd function of $t$.

$\therefore$ $I=0$.

and if $r$ is even, the integrand in $I$ is an even function of $t$.

$$\therefore\quad I=\sum_{s=0}^{\infty}\frac{(ix)^{2s}}{2s\,!}.2\int_0^1(1-t^2)^{n-1/2}\,t^{2s}\,dt$$

$$=\sum_{s=0}^{\infty}\frac{(ix)^{2s}}{2s\,!}\int_0^1(1-v)^{n-1/2}\,.v^{s-1/2}\,dv$$

Putting $t^2=v$ so that $2t\,dt=dv$

$$=\sum_{s=0}^{\infty}\frac{(ix)^{2s}}{2s\,!}.B\,(n+\tfrac{1}{2},\,s+\tfrac{1}{2}),$$

since $B\,(m,n)=\int_0^1(1-t)^{m-1}\,t^{n-1}\,dt$

$$=\sum_{s=0}^{\infty}.\frac{(ix)^{2s}}{2s\,!}\frac{\Gamma(n+\frac{1}{2})\;\Gamma(s+\frac{1}{2})}{\Gamma(n+\frac{1}{2}+s+\frac{1}{2})}\;(n>-\tfrac{1}{2})$$

since $B\,(m,n)=\dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

$$=\Gamma(n+\tfrac{1}{2})\sum_{s=0}^{\infty}\frac{(i^2)^s\,x^{2s}.\{(2s+1)/2\}}{2s\,!\,\Gamma(n+s+1)}$$

$$=\Gamma(n+\tfrac{1}{2}).\sum_{s=0}^{\infty}\frac{(-1)^s\,.\,x^{2s}}{2s\,!.\Gamma(n+s+1)}\cdot\frac{(2s)\,!}{2^{2s}\,s\,!}\sqrt{\pi},$$

since $\Gamma\left(\dfrac{2s+1}{2}\right)=\dfrac{(2s)\,!\,\sqrt{\pi}}{2^{2s}\,s\,!}$.

$$=\Gamma(n+\tfrac{1}{2}).\left(\frac{x}{2}\right)^{-n}\sum_{s=0}^{\infty}\frac{(-1)^s}{s\,!\,\Gamma(n+s+1)}\cdot\left(\frac{x}{2}\right)^{n+2s}.\sqrt{\pi}$$

$$=\Gamma(n+\tfrac{1}{2}).\left(\frac{x}{2}\right)^{-n}.\sqrt{\pi}\,.J_n(x)$$

$$\therefore\quad J_n(x)=\frac{(x/2)^n}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_{-1}^{+1}(1-t^2)^{n-1/2}\,e^{ixt}\,dt \qquad \ldots(1)$$

**Proved.**

**Deduction.**

From (1), we have $J_n(x)=\dfrac{x^n}{2^n\,\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})}\cdot$

$$\int_{-1}^{1}(1-t^2)^{n-1/2}\,(\cos xt+i\sin xt)\,dt$$

Equating real parts from both sides, we get

$$J_n(x)=\frac{x^n}{2^n\,\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{n-1/2}\cos(xt)\,dt$$

$$=\frac{x^n}{2^n\,\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})}.2\int_0^1(1-t^2)^{n-1/2}\cos(xt)\,dt$$

or $\displaystyle\int_0^1 (1-t^2)^{n-1/2} \cos (xt)\, dt = \frac{2^{n-1}\,\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})\,J_n(x)}{x^n}$

**Ex. 20.** *Prove that*

$$J_n(x)=(-2)^n . x^n \frac{d^n}{d(x^2)^n} J_0(x).$$

**[Agra 80; Meerut 79 (S); Raj. 78]**

**Proof.** Bessel's equation for zeroeth order is

$$y_2+\frac{1}{x}\, y_1+y=0 \qquad \ldots(1)$$

whose solution is $J_0(x)$.

Changing the independent variable from $x$ to $X$, by the relation $x^2=X$, so that

$$y_1=\frac{dy}{dx}=\frac{dy}{dX}\cdot\frac{dX}{dx}=2x\,\frac{dy}{dX}=2\sqrt{X}\,\frac{dy}{dX}$$

and
$$y_2=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(2\sqrt{X}\,\frac{dy}{dX}\right)$$
$$=\frac{d}{dX}\left(2\sqrt{X}\,\frac{dy}{dX}\right)\cdot\frac{dX}{dx}$$
$$=\left(2\sqrt{X}\,\frac{d^2y}{dX^2}+\frac{1}{\sqrt{X}}\frac{dy}{dX}\right).2\sqrt{X}$$
$$=4X\frac{d^2y}{dX^2}+2\,\frac{dy}{dX}$$

Substituting in (1), we have

$$\left(4X\,\frac{d^2y}{dX^2}+2\,\frac{dy}{dX}\right)+\frac{1}{\sqrt{X}}\,2\sqrt{X}.\frac{dy}{dX}+y=0,$$

or
$$4X\,\frac{d^2y}{dX^2}+4\,\frac{dy}{dX}+y=0. \qquad \ldots(4)$$

Differentiating (2) $n$ times w.r.t. $X$, by Leibnit's theorem we have

$$4\left[X\,\frac{d^{n+2}\,y}{dX^{n+2}}+n.1.\frac{d^{n+1}\,y}{dX^{n+1}}\right]+4\,\frac{d^{n+1}\,y}{dX^{n+1}}+\frac{d^ny}{dX^n}=0$$

or
$$4X\frac{d^{n+2}\,y}{dX^{n+2}}+4.(n+1)\,\frac{d^{n+1}\,y}{dX^{n+1}}+\frac{d^ny}{dX^n}=0$$

Putting $Y=\dfrac{d^ny}{dX^n}=\dfrac{d^nJ_0(x)}{dX^n}$, it becomes

$$4X\frac{d^2Y}{dX^2}+4\,(n+1)\,\frac{dY}{dX}+Y=0. \qquad \ldots(3)$$

Again, $J_n(x)$ is the solution of the Bessel's equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0 \qquad \ldots(4)$$

Putting $y=x^nZ$,

so that $\dfrac{dy}{dx}=x^n\dfrac{dZ}{dx}+nx^{n-1}Z$

and $\dfrac{d^2y}{dx^2}=x^n\dfrac{d^2Z}{dx^2}+2nx^{n-1}\dfrac{dZ}{dx}+n(n-1)x^{n-2}Z=0.$

Substituting in (4), we have

$$x^n\frac{d^2Z}{dx^2}+2nx^{n-1}\frac{dZ}{dx}+n(n-1)x^{n-2}Z$$

$$+\frac{1}{x}\left[x^n\frac{dZ}{dx}+nx^{n-1}Z\right]+\left[1-\frac{n^2}{x^2}\right]x^nZ=0$$

or $$x^n\frac{d^2Z}{dx^2}+(2n+1)x^{n-1}\frac{dZ}{dx}+x^nZ=0$$

or $$\frac{d^2Z}{dx^2}+(2n+1)\frac{1}{x}\frac{dZ}{dx}+Z=0.$$

Putting $x^2=X$, it becomes

$$\left[4X\frac{d^2Z}{dX^2}+2\frac{dZ}{dX}\right]+(2n+1).\frac{1}{\sqrt{X}}.2\sqrt{X}\frac{dZ}{dX}+Z=0.$$

or $$4X\frac{d^2Z}{dX^2}+n(n+1)\frac{dZ}{dX}+Z=0. \qquad ...(5)$$

Comparing (3) and (5), we get

$$Z=Y=\frac{d^nJ_0(x)}{dX^n}=\frac{d^nJ_0(x)}{d(x^2)^n}.$$

But $y=x^nZ$.

Hence $$J_n(x)=cx^n\frac{d^nJ_0(x)}{d(x^2)^n} \qquad ...(6)$$

where $c$ is a constant to be determined.

We know that $J_0(x)=\sum_{r=0}^{\infty}\dfrac{(-1)^r x^{2r}}{(2^r.r!)^2}$

$$\therefore \frac{d^nJ_0(x)}{d(x^2)^n}=\frac{d^n}{d(x^2)^n}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2^r.r!)^2}$$

$$=\frac{d^n}{d(x^2)^n}\sum_{r=0}^{\infty}\frac{(-1)^{n+r}(x^2)^{n+r}}{(2^{n+r}(n+r)!)^2}$$

[Since all those terms in which the index of $x$ is less than $2n$ will vanish on differentiation $n$ times w.r.t. $x^2$].

$$=\sum_{r=0}^{\infty}\frac{(-1)^{n+r}\,(n+r)\,(n+r-1)\ldots(r+1)\,(x^2)^r}{(2^{n+r}.(n+r)\,!)^2}$$

$$=(-1)^n\sum_{r=0}^{\infty}(-1)^r.\frac{(n+r)\,!}{r\,!.2^{2(n+r)}.[(n+r)\,!]^2}.x^{2r}$$

$$\therefore\quad J_n(x)=cx^n.(-1)^n\sum_{r=0}^{\infty}(-1)^r\frac{1}{r\,!.2^{2n+2r}.(n+r)\,!}.x^{2r}$$

$$=\frac{c\,(-1)^r}{2^n}\sum_{r=0}^{\infty}(-r)^r.\frac{1}{r\,!\,\Gamma\,(n+r+1)}.\left[\frac{x}{2}\right]^{n+2r}$$

$$=\frac{c}{(-2)^n}.J_n(x).$$

$$\therefore\quad c=(-2)^n$$

Hence $J_n(x)=(-2)^n.x^n.\dfrac{d^n.J_0(x)}{d\,(x^2)^n}$.

**Proved.**

**Ex. 21.** *Prove that $J_n(x)=0$ has no repeated roots except at* $x=0$.

**Proof.** If possible let $\alpha$ be a repeated root of $J_n(x)=0$.

$$\therefore\quad J_n(\alpha)=0 \text{ and } J_n'(\alpha)=0. \qquad \ldots(1)$$

From Recurrence formula I and II, we have

$$J_{n+1}\,(x)=\frac{n}{x}\,J_n-J_n'(x)$$

and
$$J_{n-1}\,(x)=\frac{n}{x}\,J_n+J_n'(x).$$

$\therefore$ $J_{n+1}\,(\alpha)=0$ and $J_{n-1}\,(\alpha)=0$ with the help of (1), *i.e.*, for the same value of $x$, $J_n(x)$, $J_{n+1}\,(x)$ and $J_{n-1}\,(x)$ are all zero, which is absurd as we cannot have two power series having the same sum function. Thus $J_n(x)=0$ cannot have repeated roots except at $x=0$.

**Ex. 22.** *Prove*

$$\int_0^{\infty}e^{-ax}\,J_0\,(bx)\,dx=\frac{1}{\sqrt{(a^2+b^2)}},\ a>0.$$

**[Rohilkhand 85; Meerut 86, 90; Raj. 78]**

**Proof.** From Ex. 18 (*b*), we have

$$J_0(x)=\frac{1}{\pi}\int_0^{\pi}\cos\,(x\,\sin\,\phi)\,dx.$$

$$\therefore\quad\int_0^{\infty}e^{-ax}\,J_0(bx)\,dx=\int_0^{\infty}e^{-ax}\left\{\frac{1}{\pi}\int_0^{\pi}\cos\,(bx\,\sin\,\phi)\,d\phi\right\}dx$$

$$=\frac{1}{\pi}\int_0^\pi\left[\int_0^\infty e^{-ax}\cos(bx\sin\phi)\,dx\right]d\phi$$

$$=\frac{1}{\pi}\int_0^\pi\left[\int_0^\infty e^{ax}.\frac{e^{i(bx\sin\phi)}+e^{-i(bx\sin\phi)}}{2}\,dx\right]d\phi$$

$$=\frac{1}{2\pi}\int_0^\pi\left[\int_0^\infty\{e^{-(a-ib\sin\phi)x}+e^{-(a+ib\sin\phi)x}\}\,dx\right]d\phi$$

$$=\frac{1}{2\pi}\int_0^\pi\left[\frac{e^{-(a-ib\sin\phi)x}}{-(a-ib\sin\phi)}-\frac{e^{-(a+ib\sin\phi)x}}{a+ib\sin\phi}\right]_0^\infty d\phi$$

$$=\frac{1}{2\pi}\int_0^\pi\left[\frac{1}{a-ib\sin\phi}+\frac{1}{a+ib\sin\phi}\right]d\phi$$

$$=\frac{1}{2\pi}\int_0^\pi\frac{2a\,d\phi}{a^2+b^2\sin^2\phi}$$

$$=2.\frac{a}{\pi}\int_0^{\pi/2}\frac{\operatorname{cosec}^2\phi\,d\phi}{b^2+a^2\operatorname{cosec}^2\phi}$$

$$=2.\frac{a}{\pi}\int_0^{\pi/2}\frac{\operatorname{cosec}^2\phi\,d\phi}{(a^2+b^2)+a^2\cot^2\phi}$$

$$=2.\frac{a}{\pi}\left(\frac{1}{a\sqrt{(a^2+b^2)}}\cot^{-1}\frac{a\cot\phi}{\sqrt{(a^2+b^2)}}\right)_0^{\pi/2}$$

$$=\frac{2}{\pi\sqrt{(a^2+b^2)}}.(\cot^{-1}0-\cot^{-1}\infty)=\frac{1}{\sqrt{(a+b^2)}}$$

**Proved.**

**Ex. 23.** *Prove that*

$$\int_0^x t\,\{J_n(t)\}^2\,dt=\frac{x^2}{2}\,\{J_n^2(x)-J_{n-1}(x)\,J_{n+1}(x)\}.$$

**Proof.** We have

$$\frac{d}{dt}\left[\frac{t^2}{2}\{J_n^2(t)-J_{n-1}(t)\,J_{n+1}(t)\}\right]$$

$$=t\,(J_n^2-J_{n-1}J_{n+1})+\frac{t^2}{2}\,[2J_nJ_n'-J'_{n-1}J_{n+1}-J_{n-1}J'_{n+1}]$$

(writing $J_n(t)$ as $J_n$ in short)

$$=t\,(J_n^2-J_{n-1}\,J_{n+1})+\frac{t^2}{2}\left[J_n\,(J_{n-1}-J_{n+1})\right.$$

$$\left.-\left(\frac{n-1}{t}J_{n-1}-J_n\right)J_{n+1}-J_{n-1}\left(-\frac{n+1}{t}J_{n+1}+J_n\right)\right]$$

(Substituting the value of $2J_n'$ from recurrence formula III and the value of $2J'_{n-1}$ and $J'_{n+1}$ obtained by replacing $n$ by $n-1$ and $n+1$ in recurrence formula I and II respectively.

$$\frac{d}{dt}\left[\frac{t^2}{2}\left\{J_n^2(t)-J_{n-1}(t)\,J_{n+1}(t)\right\}\right]=t\,J_n^2(t)$$

Integrating both sides from 0 to $x$, we have

$$\int_0^x t\,\{J_n(t)\}^2\,dt=\left[\frac{t^2}{2}\{J_n^2(t)-J_{n-1}(t)\,J_{n+1}(t)\}\right]_0^x$$

$$=\frac{x^2}{2}\{J_n^2(x)-J_{n-1}(x)\,J_{n+1}(x)\}.$$ **Proved.**

## § 5·8. A second solution of Bessel's Equation.

We see that when $n$ is zero, the two solutions $J_n(x)$ and $J_{-n}(x)$ of the Bessel's equation are equal.

The Bessel's equation for $n=0$ is

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0. \qquad ...(1)$$

This is an equation of the second order, and hence it must have two solutions. Since the two roots of $k^2=0$ in (3) of § 5·4 are identical, each being equal to zero, we get only one solution $J_0(x)$ of (1) in § 5·4. Therefore to obtain the general solution of (1) another independent solution should be found out.

For this we proceed as follows.

Let $y_0=uJ_0(x)+v$ be the solution of (1), $u$ and $v$ are functions of $x$.

Substituting in (1), we have

$$\left[u\frac{d^2J_0}{dx^2}+2\frac{du}{dx}\cdot\frac{dJ_0}{dx}+\frac{d^2u}{dx^2}J_0+\frac{d^2v}{dx^2}\right]$$
$$+\frac{1}{x}\left[u\frac{dJ_0}{dx}+\frac{du}{dx}J_0+\frac{dv}{dx}\right]+nJ_0+v=0$$

or
$$\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}+v=-J_0\left[\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}\right]-2\frac{du}{dx}\frac{dJ_0}{dx}$$
$$-u\left[\frac{d^2J_0}{dx^2}+\frac{1}{x}\frac{dJ_0}{dx}+J_0\right]$$

or
$$\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}+v=-J_0\left[\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}\right]-2\frac{du}{dx}\cdot\frac{dJ_0}{dx} \qquad ...(2)$$

Since $\dfrac{d^2J_0}{dx^2}+\dfrac{1}{x}\dfrac{dJ_0}{dx}+J_0=0$

as $J_0$ is the solution of (1).

Now $u$ being at our choice, to make equation simpler we take

$$u=\log x \text{ so that } \frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}=0.$$

$\therefore$ From (2), we have

$$\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}+v=-2.\ \frac{1}{x}\frac{dJ_0}{dx}$$

$$=\frac{2}{x}(J_1). \qquad \text{Since } J_0'=-J_1 \text{ see Ex. 2.}$$

$$=\frac{2}{x}\left[\frac{2}{x}\{2J_2-4J_4+6J_6\ldots\}\right]$$

[From Ex. 5 deduction p. 110 putting $n=1$]

$$=\frac{4}{x^2}[2J_2-4J_4+6J_6\ldots]$$

$$=\frac{4}{x^2}\Sigma(-1)^{n/2-1}.nJ_n \qquad \ldots(3)$$

when $n$ is even.

Now $\lambda J_n$ is the solution of the equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=\lambda.\ \frac{n^2}{x^2}J_n. \qquad \ldots(4)$$

[ Since substituting $y=\lambda J_n$ in (4), we get

$$\lambda\frac{d^2J_n}{dx^2}+\frac{1}{x}\lambda\frac{dJ_n}{dx}+\lambda J_n-\lambda\frac{n^2}{x^2}J_n=0$$

or
$$\frac{d^2J_n}{dx^2}+\frac{1}{x}\frac{dJ_n}{dx}+\left(1-\frac{n^2}{x^2}\right)J_n=0$$

which is correct since $J_n$ is the solution of Bessel's eqn.

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0 \quad ]$$

$\therefore$ From (3),

$$\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}+v=\frac{4}{x^2}\Sigma(-1)^{n/2-1}.nJ_n$$

$$=\frac{1}{x^2}\Sigma n^2.\frac{4(-1)^{n/2-1}}{n}J_n$$

$$=\Sigma\frac{n^2}{x^2}.\frac{4(-1)^{n/2-1}}{n}J_n. \qquad \ldots(5)$$

The general term on the R.H.S. of (5) is

$$\frac{n^2}{x^2}.\frac{4(-1)^{n/2-1}}{n}J_n.$$

$\therefore$ Comparing with (4), we have

$$\lambda_n=(-1)^{n/2-1}.\frac{4}{n}$$

for the general term.

$\therefore$ The solution of (5) is

$$v=\Sigma\ \lambda_n J_n \text{ [similarly as } \lambda J_n \text{ is the solution of (4)]}$$

$$=\Sigma\ (-1)^{n/2-1}\cdot\frac{4}{n}\ J_n,\ n \text{ being even}$$

$$=2\left[J_2-\frac{J_4}{2}+\frac{J_6}{3}-\frac{J_8}{4}+\ldots\right].$$

Hence the solution of (1) is

$$y_0=uJ_0\ (x)+v$$

$$=J_0\ (x)\log x-2\left[J_2-\frac{J_4}{2}+\frac{J_6}{3}-\frac{J_8}{4}\ldots\right] \qquad \ldots(6)$$

Since $J_n=\dfrac{x^n}{2^n\ n\ !}\left[1-\dfrac{x^2}{2^2\ (n+1)}+\dfrac{x^4}{2^4\ 2\ !\ (n+1)\ (n+2)}\ldots\right]$

$$\therefore \quad J_2=\frac{x^2}{2^2\ 2\ !}\left[1-\frac{x^2}{2^2.3}+\frac{x^4}{2^2.2\ !\ 3.4}\ldots\right]$$

$$J_4=\frac{x^4}{2^4.4\ !}\left[1-\frac{x^2}{2^2.5}+\frac{x^4}{2^2.2\ !\ 5.6}\ldots\right]$$

$$J_6=\frac{x^6}{2^6.6}\left[1-\frac{x^2}{2^2.7}+\frac{x^4}{2^4.2\ !\ 7.8}\ldots\right] \text{ etc.}$$

Substituting these in (6) and simplifying, we have

$$y_0=J_0(x)\log x+2\left[\frac{x^2}{2^2.2\ !}-\frac{x^4}{2^4.3\ !}\ (1+1/8)\right.$$
$$\left.+\frac{x^6}{2^6.4\ !}\left(\frac{1}{2}+\frac{1}{10}+\frac{1}{6.5.3}\right)\ldots\right]$$

$$=J_0\ (x)\log x+\left[\frac{x^2}{2^2.(1\ !)^2}-\frac{x^4}{2^4.(2\ !)^2}\ (1+\tfrac{1}{2})\right.$$
$$\left.+\frac{x^6}{2^6.(3\ !)^2}\ (1+\tfrac{1}{2}+\tfrac{1}{3})\ldots\right]$$

$$\mathbf{y_0=J_0(x)\ \log\ (x)+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}\ b_m}{2^{2m}\ (m\ !)^2}\ x^{2m}}$$

where $\qquad b_m=1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{m}$

This solution thus obtained is known as the **Bessel function of the second kind** of order zero or **Neumann's function** of order zero and is denoted by $y_0\ (x)$.

Thus the general solution of (1), may be taken as

$$\mathbf{y=AJ_0\ (x)+By_0\ (x).}$$

**Aliter.** The general solution of the Bessel's equation for $n=0$ ; *i.e.*, for $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}-y=0.$ $\qquad \ldots(1)$

must be of the form

$$y(x)=J_0(x)\log x+\sum_{m=1}^{\infty} A_m x^m$$

so that $\dfrac{dy}{dx}=J_0'\log x+\dfrac{1}{x}J_0+\sum\limits_{m=1}^{\infty} mA_m x^{m-1}$

and $\dfrac{d^2y}{dx^2}=J_0''\log x+\dfrac{2}{x}J_0'-\dfrac{1}{x^2}J_0+\sum\limits_{m=1}^{\infty} m(m-1)A_m x^{m-2}.$

Substituting in (1), we have

$$J_0''\log x+\frac{2}{x}J_0'-\frac{1}{x^2}J_0+\sum_{m=1}^{\infty} m(m-1)A_m x^{m-2}$$

$$+\frac{J_0'}{x}\log x+\frac{1}{x^2}J_0+\sum_{m=1}^{\infty} mA_m x^{m-1}+J_0\log x+\sum_{m=1}^{\infty} A_m x^m=0$$

or $\log x.\left[\dfrac{d^2J_n}{dx^2}+\dfrac{2}{x}\dfrac{d}{dx}J_0+J_0\right]+\dfrac{2}{x}J'_0$

$$+\sum_{m=1}^{\infty} m(m-1)A_m x^{m-2}+\frac{1}{x}\sum_{m=1}^{\infty} mA_m x^{m-1}+\sum_{m=1}^{\infty} A_m x^m=0.$$

[The first bracket is zero, since $J_0$ is the solution of (1)]

or $\quad 2J'_0+\sum\limits_{m=1}^{\infty} m(m-1)A_m x^{m-1}+\sum\limits_{m=1}^{\infty} mA_m x^{m-1}+\sum\limits_{m=1}^{\infty} A_m x^{m+1}=0$

or $\quad 2\dfrac{d}{dx}\sum\limits_{m=1}^{\infty}\dfrac{(-1)^m x^{2m}}{2^{2m}(m!)^2}+\sum\limits_{m=1}^{\infty} m^2A_m x^{m-1}+\sum\limits_{m=1}^{\infty} A_m x^{m+1}=0$

or $\quad 2\sum\limits_{m=1}^{\infty}\dfrac{(-1)^m 2m\, x^{2m-1}}{2^{2m}(m!)^2}+\sum\limits_{m=1}^{\infty} m^2A_m x^{m-1}+\sum\limits_{m=1}^{\infty} A_m x^{m-1}=0$

or $\quad \sum\limits_{m=1}^{\infty}\dfrac{(-1)^m x^{2m-1}}{2^{2m-1}m!(m-1)!}+\sum\limits_{m=1}^{\infty} m^2A_m x^{m-1}+\sum\limits_{m=1}^{\infty} A_m x^{m+1}=0$

.. (2)

Since (2) is an identity, equating the coefficient of $x^0$, $A_1=0$. Equating the sum of the coefficient of $x^{2r}$ to zero we obtain

$$(2r+1)^2 A_{2r+1}+A_{2r-1}=0.$$

$$\therefore\ A_{2r+1}=\frac{1}{(2r+1)^2}.A_{2r-1}$$

Since $A_1=0$, putting $r=1, 2, 3$...etc., we have

$A_1=A_3=A_5=\ldots=0$ (each)

*i.e.* $\quad A_m$ with odd subscripts are all zero.

Again equating to zero the sum of the coefficients of $x^{2r+1}$, we have

$$\frac{(-1)^{r+1}}{2^{2r}(r+1)!\,r!}+(2r+2)^2A_{2r+2}+A_{2r}=0. \qquad ...(2)$$

Putting $r=0$, $-\frac{1}{1!}+2^2.A_2=0$. [$A_0$ does not appear in (2)]

$$\therefore \quad A_2=\frac{1}{4}=\frac{1}{2^2(1!)^2}.1$$

Also for $r=1$, $\frac{1}{2^2.2!}+4^2A_4+A_2=0$

$$\therefore \quad 16A_4=-1/4-1/8$$

$$A_4=-\frac{1}{2^4(2!)^2}(1+\tfrac{1}{2})$$

And in general

$$A_{2m}=\frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m}\right)$$

$$=\frac{(-1)^{m-1}}{2^{2m}(m!)^2}h_m \text{ if } h_m=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m}$$

$$\therefore \quad y(x)=J_0(x)\log x+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}.x^{2m}.$$

Since $A_1=A_3=A_5=\ldots=0$ (each)

$$=J_0(x)\log x+\left[\frac{1}{2^2}x^2-\frac{x^4}{2^2(2!)^2}(1+\tfrac{1}{2})+\ldots\right]$$

This is the required Bessel's function of the second kind and is denoted by $y_0(x)$.

$$\therefore \quad y_0(x)=J_0(x)\log x+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}.x^{2m}$$

where $\quad h_m=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m}.$

**Note.** A standard practice is to define the second solution of the Bessel's equation by the function

$$y_n(x)=\frac{J_n(x)\cos n\pi-J_{-n}(x)}{\sin n\pi},\ n\neq\text{integer}$$

and $\quad y_n(x)=\lim_{\nu\to n}\frac{J_n(x)\cos\nu\pi-J_{-n}(x)}{\sin n\pi}$

where $n$ is an integer.

This is known as **Bessel function of the second kind of order n.**

It can be shown that $y_n(x)$ satisfies recurrence formulae of the same type as those for $J_n(x)$.

## EXERCISE ON CHAPTER V

1. For what value of $n$ the general solution of the Bessel's differential equation will be of the form $y=AJ_n(x)+BJ_{-n}(x)$. **[Meerut 78]**

2. Prove that $y=\frac{1}{\pi}\int_0^{\pi}\cos(x\cos\phi)\,d\phi$ satisfies the Bessel's differential equation of zeroeth order.

3. Show that

(i) $J_{-3/2}(x)=-\sqrt{\left(\frac{2}{\pi x}\right)}\left[\sin x+\frac{1}{x}\cos x\right]$

(ii) $J_{5/2}(x)=\left[\frac{2}{\pi x}\right]^{1/2}\left[\frac{3-x^2}{x^2}\sin x-\frac{3}{x}\cos x\right]$ [Meerut 83 (P)]

4. Show that $\int_0^x x^n J_{n-1}(x)\,dx=x^n J_n(x)$.

[Hint. Integrate recurrence formula VI between the limits 0 to $x$].

5. Show that $\int_0^x x^{n+1} J_n(x)\,dx=x^{n+1} J_{n+1}(x)$.

[Hint. Replace $n$ by $(n+1)$ in recurrence formula VI, and then integrate between the limits 0 to $x$].

6. Show that $y=x^{-n/2} J_n(2\sqrt{x})$ satisfies the differential equation

$$x\frac{d^2y}{dx^2}+(n+1)\frac{dy}{dx}+y=0.$$

7. Show that

$$J_{2n}(x)=(-1)^n\frac{2}{\pi}\int_0^{\pi/2}\cos 2n\phi\cos(x\sin\phi)\,d\phi$$

[Hint. Put $\phi=\theta+\frac{\pi}{2}$ in $J_{2n}(x)=\frac{2}{\pi}\int_0^{\pi/2}\cos 2n\phi\cos(x\sin\phi)\,d\phi$]

8. Show that

$$J_{2n-1}(x)=-(-1)^n\frac{2}{\pi}\int_0^{\pi/2}\cos(2n+1)\,\theta\sin(x\cos\theta)\,d\theta.$$

9. Prove that

$$J_n(x)=\frac{2.(x/2)^{n-m}}{\Gamma(n-m)}\cdot\int_0^1(1-t)^{n-m-1}J_n\,t^{m+1}(xt)\,dt$$

$(n>m>-1)$.

10. Prove that

$$\int_0^{\infty}\frac{J_n(x)}{x}\,dx=\frac{1}{n}.$$

[Meerut 77, 86 (R)]

11. $\int_0^{\pi/2}\sqrt{(\pi x)}.J_{1/2}(2x)\,dx=1.$ [Meerut 80 (S)]

12. Establish the relation

$$2J'_n(x)=J_{n-1}(x)-J_{n+1}(x)$$

for the Bessel's function $J_n(x)$.

Hence deduce that

$$J''_n(x)=\tfrac{1}{4}[J_{n-2}(x)-2J_n(x)+J_{n+2}(x)].$$ (G.N.U.A. 81)

13. Prove that $\lim\limits_{x\to 0} \dfrac{J_n(x)}{x^n} = \dfrac{1}{2^n \Gamma(n+1)}$ [Raj. 79]

14. Show that

$$\int_0^1 J_0 [\sqrt{\{x (t-x)\}}]\, dx = 2 \sin \tfrac{1}{2} t.$$

[Kanpur 84]

15. Prove that $\displaystyle\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx = 0,\ \alpha \neq \beta.$

[Rohilkhand 88; Kanpur 81]

16. Prove that $\displaystyle\int_0^b xJ_0(ax)\, dx = \frac{b}{a}\, J_1(ab).$

[Rohilkhand 82; Kanpur 81]

17. Prove that

(a) $4 \displaystyle\int J_{n+1}(x)\, dx = \int J_{n-1}(x)\, dx - 2J_n(x)$ [Kanpur 88]

(b) $4 \dfrac{d^2}{dx^2} [J_n(x)] = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$ [Kanpnr 88]

18. Prove that

$$\lim_{x\to 0} \frac{J_n(x)}{x^n} = \frac{1}{2^n \Gamma(n+1)}.$$

[Agra 88]

# 6
# Hermite Polynomials

## § 6·1. Hermite Differential Equation.

The differential equation

$$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2\lambda y=0,$$

where $\lambda$ is a constant, is called Hermite's differential equation.

## § 6·2. Solution of Hermite's equation. [Agra 78]

The Hermite's equation

$$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2\lambda y=0, \qquad \text{...(i)}$$

is solved by series integration.

Let us assume $y=\sum_{r=0}^{\infty} a_r x^{k+r}$ ...(ii)

as the solution of the given equation (i)

$$\therefore \quad \frac{dy}{dx}=\sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1}$$

and

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

Substituting in (i), we have

$$\sum_{r=0}^{\infty} a_r [(k+r)(k+r-1) x^{k+r-2}-2(k+r) x^{k+r}+2\lambda x^{k+r}]=0,$$

or

$$\sum_{r=0}^{\infty} a_r [(k+r)(k+r-1) x^{k+r-2}-2(k+r-\lambda) x^{k+r}]=0 \qquad \text{...(iii)}$$

Now (iii) being an identity, we can equate to zero the coefficients of various powers of $x_r$.

$\therefore$ Equating to zero the coefficient of lowest power of $x$ *i.e.* of $x^{k-2}$, we have $a_0 k (k-1)=0$.

Now $a_0 \neq 0$, as it is the coefficient of the first term with which we start to write the series.

$$\therefore \quad \left.\begin{array}{l} \text{either } k=0 \\ \text{or} \quad k=1 \end{array}\right\}. \qquad \text{...(iv)}$$

Equating to zero the coefficient of $x^{k-1}$ in (ii), we have

$$a_1(k+1)\, k=0 \qquad \text{...(v)}$$

which implies that $a_1=0$ or $k=0$ or both are zero.

Since $k+1 \neq 0$ for any value of $k$ given by (iv).

Now equating to zero the coefficient of general term *i.e.* $x^{k+r}$ in (iii), we have

$$a_{r+2}\,(k+r+2)\,(k+r+1)-2a_r\,(k+r-\lambda)=0$$

or
$$a_{r+2}=\frac{2(k+r-\lambda)}{(k+r+2)\,(k+r+1)}\cdot a_r$$

or
$$a_{r+2}=\frac{2(k+r)-2\lambda}{(k+r+2)\,(k+r+1)}\cdot a_r. \qquad \text{...(vi)}$$

Now two cases arise.

**Case I.** When $k=0$, from (vi), we have

$$a_{r+2}=\frac{2r-2\lambda}{(r+2)\,(r+1)}\; a_r. \qquad \text{...(vii)}$$

Putting $r=0$, 2, 4, etc. in (vii), we have

$$a_2=\frac{-2\lambda}{2.1}\; a_0=-\frac{2\lambda}{2\,!}\; a_0$$

$$a_4=\frac{4-2\lambda}{4.3}\; a_2$$

$$=-\frac{(4-2\lambda)\cdot 2\lambda}{4.3.2\,!}\; a_0=\frac{2^2(-2+\lambda)\,\lambda}{4\,!}\; a_0$$

and so on

$$\therefore \quad a_{2m}=\frac{(-2)^m\,\lambda\,(\lambda-2)\ldots(\lambda-2m+2)}{(2m)\,!}\; a_0.$$

Again putting $r=1$, 3, 5, etc. in (vii), we have

$$a_3=\frac{2-2\lambda}{3.2}\; a_1=-\frac{2(\lambda-1)}{3\,!}\; a_1$$

$$a_5=\frac{6-2\lambda}{5.4}\; a_3=-\frac{2(6-2\lambda)\,(\lambda-1)}{5.4.3}\; a_1$$

$$=(-2)^2\,\frac{(\lambda-1)\,(\lambda-3)}{5\,!}\; a_1$$

and so on.

$$\therefore \quad a_{2m+1}=\frac{(-2)^m\,(\lambda-1)\,(\lambda-3)\ldots(\lambda-2m+1)}{(2m+1)\,!}\; a_1.$$

Now if $a_1 \neq 0$, then we have

$$y = \sum_{r=0}^{\infty} a_r x^r$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_0 \left[ 1 - \frac{2\lambda}{2\,!} x^2 + \frac{2^2 \lambda (\lambda-2)}{4\,!} x^4 + \ldots + \frac{(-2)^m \lambda(\lambda-2)\ldots(\lambda-2m+2)}{(2m)\,!} x^{2m} + \ldots \right]$$

$$+ a_1 \left[ x - \frac{2(\lambda-1)}{3\,!} x^3 + \frac{2^2(\lambda-1)(\lambda-3)}{5\,!} x^5 + \ldots + \frac{(-2)^m (\lambda-1)(\lambda-3)\ldots(\lambda-2m+1)}{(2m+1)\,!} x^{2m+1} + \ldots \right] \quad \ldots\text{(viii)}$$

and if $a_1 = 0$, then we have

$$y = a_0 \left[ 1 - \frac{2\lambda}{2\,!} x^2 + \frac{2^2\lambda(\lambda-2)}{4\,!} x^4 + \ldots + \frac{(-2)^m \lambda(\lambda-2)\ldots(\lambda-2m+2)}{(2m)\,!} x^{2m} + \ldots \right]$$

$$= y_1 \text{ (say)}. \quad \ldots\text{(ix)}$$

**Case II.** When $k=1$, from (vi), we have

$$a_{r+2} = \frac{2(r+1) - 2\lambda}{(r+3)(r+2)} a_r.$$

Putting $r = 1, 3 \ldots$etc.

$$a_3 = a_5 = \ldots = 0 \text{ (each)}.$$

Since in this case from (v), $a_1 = 0$.

Putting $r = 0, 2, 4$, etc.

$$a_2 = \frac{2+2\lambda}{3.2} a_0 = -\frac{2(\lambda-1)}{3\,!} a_0$$

$$a_4 = \frac{6-2\lambda}{5.4} a_2 = \frac{2(\lambda-1)(\lambda-3)}{3\,!} a_0$$

and so on.

$$\therefore \quad a_{2m} = \frac{(-2)^m (\lambda-1)(\lambda-3)\ldots(\lambda-2m+1)}{(2m+1)\,!} a_0$$

$\therefore$ We have

$$y = \sum_{r=0}^{\infty} a_r x^{r+1}$$

$$= a_0 x + a_2 x^3 + a_4 x^5 + \ldots + a_{2m} x^{2m+1} + \ldots$$

$$=a_0\left\{x-\frac{2(\lambda-1)}{3!}x^3+\frac{2^2(\lambda-1)(\lambda-3)}{5!}x^5\ldots\right.$$
$$\left.+\frac{(-2)^m(\lambda-1)(\lambda-3)\ldots(\lambda-2m+1)}{(2m-1)!}x^{2m+1}+\ldots\right\}$$
$$=y_2 \text{ (say).} \qquad \ldots(x)$$

From (viii) and (x) it is obvious that (x) is the part of solution, given by (viii). But as the two are the solutions of the same equations so (x) must not be the part of solution (viii).

$\therefore$ $a_1=0$ and the solution in the case $k=0$ must be given by (ix).

Hence the general solution of Hermite's equation is

$$y=Ay_1+By_2$$

where $A$ and $B$ are arbitrary constants and $y_1$, $y_2$ are given by (ix) and (x).

## § 6·3. Hermite's Polynomials.

[Rohilkhand 80, 83; Meerut 78, 86, 86 (R); 7odhpur 80, 85]

When $\lambda$ is an even integer, equation (ix) gives an even polynomial of degree $n$.

Let $\lambda=n$, $n$ is an even integer

and $$a_0=(-1)^{n/2}\frac{n!}{(n/2)!}.$$

$\therefore$ Coefficient of $x^n$ in (ix) is

$$(-1)^{n/2}\frac{n!}{(n/2)!}\cdot\frac{(-2)^{n/2}\,n(n-2)\ldots(n-n+2)}{n!}$$
$$=2^n.\frac{n/2\,(n/2-1)\ldots1}{(n/2)!}=2^n.$$

Similarly coefficient of $x^{n-2}$

$$=(-1)^{n/2}\frac{n!}{(n/2)!}\frac{(-2)^{(n-2)/2}n(n-2)\ldots(n-n+2+2)}{(n-2)!}$$
$$=-\frac{2^{n-2}\,n(n-1)\,n/2\,(n/2-1)\ldots2}{(n/2)!}=-\frac{n(n-1)}{1!}2^{n-2}$$

and so on.

So value of $y$ for $y=x$ is given by

$$y_n=(2x)^n-\frac{n(n-1)}{1!}(2x)^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}$$
$$\ldots+(-1)^{n/2}\frac{n!}{(n/2)!}$$

This value of $y$ is known as the **Hermite's Polynomial** of degree $n$ and is written as

$$H_n(x)=(2x)^n-\frac{n(n-1)}{1!}(2x)^{n-2}$$
$$+\frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}+\ldots+(-1)^{n/2}\frac{n!}{(n/2)!}$$

or $$H_n(x)=\sum_{r=0}^{(n/2)}(-1)^r\frac{n!}{r!(n-2r)!}(2x)^{n-2r}$$

where $\left(\frac{n}{2}\right)=\begin{cases}n/2 & \text{if } n \text{ is even}\\ \frac{1}{2}(n-1) & \text{if } n \text{ is odd}\end{cases}$

**§ 6·4. Generating function.**

**To prove** $e^{2tx-t^2}=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x).$

**[Rohilkhand 89; Meerut 80, 84 (P) ; Poona 70]**

**Proof.** Here

$$e^{2tx-t^2}=e^{2tx}\cdot e^{-t^2}$$
$$=\sum_{r=0}^{\infty}\frac{(2tx)^r}{r!}\cdot\sum_{r=0}^{\infty}\frac{(-t^2)^s}{s!}$$
$$=\sum_{r=0}^{\infty}(-1)^s\cdot\frac{(2x)^r}{r!\,s!}\cdot t^{r+2s}.$$

Coefficient of $t^n$ (for fixed value of $s$)

$$=(-1)^s\frac{(2x)^{n-2s}}{(n-2s)!\,s!}$$

Obtained by putting $r+2s=n$ i.e. $r=n-2s$

The total value of $t^n$ is obtained by summing over all allowed values of $s$, and since $r=n-2s$

$$\therefore\quad n-2s\geqslant 0 \text{ or } s\leqslant n/2.$$

Thus if $n$ is even $s$ goes from 0 to $n/2$ and if $n$ is odd, $s$ goes from 0 to $(n-1)/2$.

$$\therefore \text{ Coefficient of } t^n=\sum_{s=0}^{(n/2)}(-1)^r\frac{(2x)^{n-2s}}{(n-2s)!\,s!}=\frac{H_n(x)}{n!}$$

Hence $$e^{2xt-t^2}=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x)$$

or $$e^{x^2-(t-x)^2}=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x).$$

**Proved.**

**§ 6·5. Other forms for the Hermite Polynomials.**

**(I)** $H_n(x)=2^n\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)x^n.\right\}$ [Meerut 81; Raj. 79, 85; Jodhpur 81, 84, 86; Agra 80, 86; Rohilkhand 82]

**Proof.** We have

$$\frac{1}{2}\frac{d}{dx}e^{2tx}=te^{2tx} \qquad \ldots(1)$$

$$\frac{d}{dx}\left(\frac{1}{2}\frac{d}{dx}e^{2tx}\right)=2t^2e^{2tx}.$$

$$\therefore \quad \frac{1}{2}\frac{d}{dx}\left(\frac{1}{2}\frac{d}{dx}e^{2tx}\right)=e^{2tx}$$

or

$$\left(\frac{1}{2}\frac{d}{dx}\right)^2 e^{2tx}=t^2e^{2tx}.$$

Hence $\left(\frac{1}{2}\frac{d}{dx}\right)^n e^{2tx}=t^n e^{2tx}.$ ...(2)

Thus $\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}e^{2tx}.$

$$=\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)^n\right]e^{2tx}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{1}{2}\frac{d}{dx}\right)^{2n}e^{2tx}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n}e^{2tx} \text{ [from (2)]}$$

$$=e^{2tx}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n}=e^{2tx}\sum_{n=0}^{\infty}\frac{1}{n!}(-t^2)^n$$

$$=e^{2tx}.e^{-t^2}=e^{-t^2+2tx}$$

or $\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}\sum_{n=0}^{\infty}\frac{1}{n!}(2tx)^n=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x)$ [From § 6·4]

Equating the coefficient of $t^n$ from the two sides, we have

$$\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}\frac{1}{n!}2^nx^n=\frac{1}{n!}H_n(x)$$

$$H_n(x)=2^n\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}x^n.$$

**Proved.**

**(II) The Rodrigues Formula for $H_n(x)$.**

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}).$$

[Meerut 72, 74, 77(S), 78, 80, (S), 81 (P), 84, 86 (R), 87 ; Raj. 82, 84, 86 ; Jodhpur 82 ; Poona 70 ; B.H.U. 72]

**Proof.** We have $e^{2tx-t^2}=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n$

or $e^{x^2-(t-x)^2}=\frac{H_0(x)}{0!}t^0+\frac{H_1(x)}{1!}t+\frac{H_2(x)}{2!}t^2$

$$+\ldots+\frac{H_n(x)}{n!}t^n+\frac{H_{n+1}(x)}{(n+1)!}t^{n+1}+\ldots$$

Differentiating both sides, partially w.r.t. '$t$', $n$ times and then putting $t=0$, we have

$$\frac{H_n(x)}{n!}n!=\left[\frac{\partial^n}{\partial t^n}e^{-(t-x)^2}\right]_{t=0}e^{x^2}$$

Now let $t-x=u$ *i.e.* at $t=0$, $x=-u$.

$$\therefore\quad \frac{\partial}{\partial t}\equiv\frac{\partial}{\partial u}$$

or $\left[\frac{\partial^n}{\partial t^n}e^{-(t-x)^2}\right]_{t=0}=\frac{\partial^n}{\partial u^n}(e^{-u^2})=(-1)^n\frac{\partial^n}{\partial x^n}(e^{-x^2})$

$$=(-1)^n\frac{d^n}{dx^n}(e^{-x^2})$$

$$\therefore\quad H_n(x)=(-1)^n.e^{x^2}.\frac{d^n}{dx^n}(e^{-x^2}).$$

**§ 6·6. To find first few Hermite Polynomials,**

Puttting $n=0, 1, 2, 3, \ldots$ in $H_n(x)=(-1)^n e^{x^2}.\frac{d^n}{dx^n}(e^{-x^2})$

we have

$$H_0(x)=e^{x^2}e^{-x^2}=1$$

$$H_1(x)=(-1)\,e^{x^2}\frac{d}{dx}(e^{-x^2})=2x$$

$$H_2(x)=(-1)^2e^{x^2}\frac{d^2}{dx^2}(e^{-x^2})=e^{x^2}\frac{d}{dx}(-2x\,e^{-x^2})$$

$$=e^{x^2}(4x^2e^{-x^2}-2e^{-x^2})=e^{x^2}\{(4x^2-2)\,e^{-x^2}\}=4x^2-2$$

$$H_3(x)=(-1)^3e^{x^2}\frac{d^3}{dx^3}(e^{-x^2})=-e^{x^2}\frac{d}{dx}\left\{\frac{d^2}{dx^2}(e^{-x^2})\right\}$$

$$=-e^{x^2}\frac{d}{dx}\{(4x^2-2)\,e^{-x^2}\}$$

$$=-e^{x^2}\{-2x\,(4x^2-2)\,e^{-x^2}+8x\,e^{-x^2}$$

$$=-e^{x^2}\{(-8x^3+12x)\,e^{-x^2}\}=8x^3-12x.$$

Similarly $H_4(x)=16x^4-48x^2+12$

$H_5(x)=32x^5-160x^3+120x$ etc. [Meerut 86]

## § 6·7. Orthogonal Properties of Hermite Polynomials.

**To prove that**

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\, H_m(x)\, dx=\begin{cases} 0 & \text{if } m\neq n \\ \sqrt{\pi}\,2^n\,(n)\,! & \text{if } m=n \end{cases}$$

**[Agra 76; Meerut 73, 77, 80, 82 (P), 83 (P), 84, 85, 87, 88, 90; B.H.U. 72 ; Jodhpur 82 ; Raj. 77, 79, 81, 82, 84; Rohilkhand 84]**

**Proof.** We have $e^{-t^2+2tx}=\sum_{n=0}^{\infty} H_n(x)\dfrac{t^n}{n\,!}$

and $e^{-s^2+2sx}\sum_{m=0}^{\infty} H_m(x)\dfrac{s^m}{m\,!}$

$$\therefore\quad e^{-t^2+2tx}\, e^{-s^2+2sx}=\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n\,!}\sum_{m=0}^{\infty} H_m(x)\frac{s^m}{m\,!}$$

$\therefore\quad \dfrac{1}{n\,!\,m\,!} H_n(x)\, H_m(x)=$coeff. of $t^n s^m$ in the expansion of

$$e^{-t^2+2tx}\cdot e^{-s^2+2sx}$$

$\therefore\quad \int_{-\infty}^{\infty} e^{-x^2} H_n(x)\, H_m(x)\, dx$ is $n\,!\,m\,!$ times the coefficient of $t^n\, s^m$ in the expansion of $\int_{-\infty}^{\infty} e^{-x^2}\, e^{-t^2+2tx}\, e^{-s^2+2sx}\, dx$

Now $\int_{-\infty}^{\infty} e^{-x^2}\, e^{-t^2+2tx}\, e^{-s^2+2sx}\, dx$

$$=e^{-t^2-s^2}\int_{-\infty}^{\infty} e^{-x^2+2tx+2sx}\, dx$$

$$=e^{-t^2-s^2}\int_{-\infty}^{\infty} e^{-[x^2-(t+s)]^2+(t+s)^2]}\, dx$$

$$=e^{2ts}\int_{-\infty}^{\infty} e^{-[x-(t+s)]^2}\, dx$$

$=e^{2ts}\int_{-\infty}^{\infty} e^{-u^2}\, du.$ Putting $x-(t+s)=u$

$=e^{2ts}\sqrt{\pi}.$ Since $\int_{-\infty}^{\infty} e^{-u^2}\, du=\pi$

$$=\sqrt{\pi}\left[1+2ts+\frac{(2ts)^2}{2!}+\ldots+\frac{(2ts)^n}{n!}+\ldots\right]$$

Coefficient of $t^n s^m$ in the expansion of

$$\int_{-\infty}^{\infty} e^{-x^2} e^{-t^2+2tx} e^{-s^2+2sx} dx$$

is $\quad 0 \quad$ if $m \neq n$

and $\quad \dfrac{2^n\sqrt{\pi}}{n!} \quad$ if $m=n$.

Hence

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\, H_m(x)\, dx=\begin{cases}0 & \text{if } m\neq n\\ \sqrt{\pi}\, 2^n\, n! & \text{if } m=n\end{cases}$$

**Note.** Marking use of the Kronecker delta, we have

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\, H_m(x)\, dx=\sqrt{\pi}\, 2^n\, n!\, \delta_{mn},\ \delta_{mn}=\begin{cases}0 \text{ if } m\neq n\\ 1 \text{ if } m=n\end{cases}$$

**§ 6·8. Recurrence formulae for Hermite Polynomials.**

**(I) $H_n'(x)=2n\, H_{n-1}(x)$ $n\geqslant 1$.**

**[Meerut 74 (S), 75, 78, 81 (P), 83; Poona 70; Raj. 83; Agra 85; Rohilkhand 83, 85]**

**Proof.** We have

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{-t^2+2tx}$$

Differentiating both sides w.r.t. '$x$', we have

$$\sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^n = e^{-t^2+2xt}\cdot 2t$$

$$=2t\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

$$=2\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1}$$

$$=2\sum_{n=1}^{\infty} \frac{H_{n-1}(x)}{(n-1)!} t^n.$$

Equating the coefficient of $t^n$, on both the sides, we have

$$\frac{H_n'(x)}{n!}=2\,\frac{H_{n-1}(x)}{(n-1)}$$

i.e. $\quad H_n'(x)=2n\, H_{n-1}(x)$. **Proved.**

**(II) $2x\, H_n(x)=2n\, H_{n-1}(x)+H_{n+1}(x)$**

**[Agra 77, 82; Meerut 71, 73 (S), 79, 81, 83, 85, 86, 89; B.H.U. 72; Rohilkhand 83, 85, 86, 88; Raj. 78, 81, 85]**

**Proof.**

We have $\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{-t^2+2tx}$.

Differentiating both sides w.r.t. '$t$' we have

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} nt^{n-1} = e^{-t^2+2tx}.(-2t+2x)$$

or $$\sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)} t^{n-1} = -2te^{-t^2+2tx} + 2xe^{-t^2+2tx}$$

or $$\sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1} = -2t \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n + 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

(Since term of L.H.S. corresponding to $n=0$ is zero)

or $$2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} + \sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1}$$

or $$2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = 2 \sum_{n=1}^{\infty} \frac{H_{n-1}(x)}{(n-1)!} t^n + \sum_{n=0}^{\infty} \frac{H_{n+1}(x)}{n!} t^n.$$

Equating the coefficients of $t^n$, on the two sides, we have

$$2x \frac{H_n(x)}{n!} = 2 \frac{H_{n-1}(x)}{(n-1)!} + \frac{H_{n+1}(x)}{n!}$$

or $$\mathbf{2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x).}$$ **Proved.**

**Note.** Equating the coefficients of $t$, we have $2xH_0(x) = H_1(x)$.

**(III) $H'_n(x) = 2xH_n(x) - H_{n+1}(x)$. [Jodhpur 84; Meerut 81(P)]**

**Proof.** Writing recurrence formulae I and II, we have

$$H'_n(x) = 2n\, H_{n-1}(x) \qquad \text{...(i)}$$

and $$2xH_n(x) = 2n\, H_{n-1}(x) + H_{n+1}(x). \qquad \text{...(ii)}$$

Subtracting (ii) from (i), we have

*i.e.* $$\mathbf{H'_n(x) = 2x\, H_n(x) - H_{n+1}(x).}$$ **Proved.**

**(IV) $H_n''(x) - 2x\, H'_n(x) + 2nH(x) = 0$. [Meerut 83, 87]**

**Proof.** Hermite's differential equation is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0. \qquad \text{...(i)}$$

$\therefore$ $H_n(x)$ is the solution of (i), therefore, we have

$$\mathbf{H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0.}$$ **Proved.**

## EXAMPLES

**Ex. 1.** *Prove that* $H'_n = 4n(n-1) H_{n-2}$.
**[Raj. 86; Jodhpur 82, 84]**

**Sol.** From recurrence formula I, we have

$$H'_n = 2nH_{n-1} \qquad \text{...(i)}$$

Differentiating w.r.t. '$x$' we have

$$H''_n = 2nH'_{n-1} \quad \text{...(ii)}$$

Replacing $n$ by $(n-1)$ in (i), we have

$$H'_{n-1} = 2\,(n-1)\,H_{n-2} \quad \text{...(iii)}$$

$\therefore$ from (ii) and (iii); we have

$$H''_n = 4n\,(n-1)\,H_{n-2}. \qquad \textbf{Proved.}$$

**Ex. 2.** *Evaluate*

$$\int_{-\infty}^{\infty} xe^{-x^2} H_n(x)\,H_m(x)\,dx.$$

[Meerut 83(P), 89; Rohilkhand 82)

**Sol.** From recurrence formula II, we have

$$xH_n(x) = nH_{n-1}(x) + \tfrac{1}{2}H_{n+1}(x)$$

$$\therefore \int_{-\infty}^{\infty} e^{-x^2} H_n(x)\,H_m(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2} \{nH_{n-1}(x) + \tfrac{1}{2}H_{n+1}(x)\}\,H_m(x)\,dx$$

$$= n\int_{-\infty}^{\infty} e^{-x^2} H_{n-1}(x)\,H_m(x)\,dx + \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} H_{n+1}(x)\,H_m(x)\,dx$$

$$= n\sqrt{\pi}\,2^{n-1}\,(n-1)\,!\;\delta_{n-1,m} + \tfrac{1}{2}\sqrt{\pi}\,2^{n+1}\,(n+1)\,!\;\delta_{n+1,m}$$

$$= \sqrt{\pi}\,2^{n-1}\,n\,!\;\delta_{n-1,m} + \sqrt{\pi}\,2^{n}\,(n+1)\,!\;\delta_{n+1,m}$$

where $\delta$ is Kronecker delta.

**Ex. 3.** *Prove that (i)* $H_{2n}(0) = (-1)^n \dfrac{(2n)\,!}{n\,!}$. [Agra 72, 82]

*and (ii)* $H_{2n+1}(0) = 0$.

**Sol.** We have

$$\sum_{n=0}^{\infty} \frac{t^n}{n\,!} H_n(x) = e^{-t^2+2tx} \quad \text{from § 6·4.}$$

Putting $x = 0$

$$\sum_{n=0}^{\infty} \frac{t^n}{n\,!} H_n(0) = e^{-t^2}$$

$$= \left[1 - t^2 + \frac{(t^2)^2}{2\,!} + \ldots + (-1)^n \frac{(t^2)^n}{n\,!} + \ldots\right] \quad \text{...(1)}$$

(i) Equating the coefficients of $t^{2n}$, on the two sides, we have

$$\frac{1}{(2n)\,!} H_{2n}(0) = (-1)^n \frac{1}{n\,!}$$

$$\therefore \quad H_{2n}(0) = (-1)^n \frac{(2n)\,!}{n\,!}$$

(ii) Again equating the coefficients of $t^{2n+1}$, on the two sides of (1), we have

$$\frac{1}{(2n+1)\,!} H_{2n+1}(0)=0.$$

(Since R.H.S. of (1) do not involve, odd powers of $t$).

$\therefore \quad H_{2n+1}(0)=0.$ **Proved.**

**Ex. 4.** *Prove that, if* $m<n$

$$\frac{d^m}{dx^m}\{H_n(x)\}=\frac{2^m\, n\,!}{(n-m)\,!} H_{n-m}(x).$$

**[Meerut 76, 79(S), 82, 84(P), 89; Agra 80, 83, 87; Raj. 78, 86; Jodhpur 82, 84]**

**Sol.** We have

$$\sum_{n=0}^{\infty} \frac{t^n}{n\,!} H_n(x)=e^{-t^2+2tx} \qquad \text{...(i)}$$

$$\therefore \quad \sum_{n=0}^{\infty} \frac{t^n}{n\,!} \cdot \frac{d^m}{dx^m}\{H_n(x)\}=\frac{d^m}{dx^m}\, e^{-t^2+2tx}$$

$$=(2t)^m . e^{-t^2+2tx}$$

$$=(2t)^m \sum_{n=0}^{\infty} \frac{t^n}{n\,!} H_n(x) \text{ from (i)}=2^m \sum_{n=0}^{\infty} \frac{1}{n\,!} t^{n+m} H_n(x)$$

$$=2^m \,.\, \sum_{r=m}^{\infty} \frac{1}{(r-m)\,!} t^r H_{r-m}(x). \text{ Putting } n+m=r,\ n=r-m$$

For $n=0$; $r=m$, for $n=\infty$, $r=\infty$.

Equating the coefficient of $t^n$ from the two sides, we have

$$\frac{1}{n\,!}\frac{d^m}{dx^m}\{H_n(x)\}=2^m \cdot \frac{1}{(n-m)\,!} H_{n-m}(x)$$

$$\therefore \quad \frac{d^m}{dx^m}\{H_n(x)\}=\frac{2^m\, n\,!}{(n-m)\,!} H_{n-m}(x).$$ **Proved.**

**Aliter.** From recurrence formula I, we have

$$H'_n(x)=2n\, H_{n-1}(x). \qquad \text{...(i)}$$

By repeated application of (i) $m$, times and proceeding as in Ex. 1, we shall get the required result.

**Ex. 5.** *Prove that* $P_n(x)=\dfrac{2}{\sqrt{\pi}\, n\,!}\displaystyle\int_0^{\infty} t^n e^{-t^2} H_n(xt)\, dt$

**(Agra 76; Jodhpur 85)**

**Sol.** We have $H_n(x)=\displaystyle\sum_{r=0}^{(n/2)} (-1)^r \frac{n\,!}{r\,!\,(n-2r)\,!} (2x)^{n-2r}$

$$\therefore \quad H_n(xt)=\sum_{r=0}^{(n/2)} (-1)^r \frac{n\,!}{r\,!\,(n-2r)\,!} (2xt)^{n-2r}$$

$$\therefore \quad \frac{2}{\sqrt{\pi}\, n\,!} \int_0^\infty t^n e^{-t^2} H_n(xt)\, dt$$

$$= \frac{2}{\sqrt{\pi}\, n\,!} \int_0^\infty t^n e^{-t^2} \sum_{r=0}^{(n/2)} (-1)^r \frac{n\,!}{r\,!\,(n-2r)\,!} (2xt)^{n-2r}\, dt$$

$$= \sum_{r=0}^{(n/2)} \frac{2^{n-2r+1}\, (-1)^r\, x^{n-2r}}{\sqrt{\pi}\, r\,!\,(n-2r)\,!} \int_0^\infty e^{-t^2}\, t^{2n-2r}\, dt$$

$$= \sum_{r=0}^{(n/2)} \frac{2^{n-2r+1}\, (-1)^r\, x^{n-2r}}{\sqrt{\pi}\, r\,!\,(n-2r)\,!} \int_0^\infty e^{-t^2} . t^{2(n+r+1/2)-1}\, dt$$

$$= \sum_{r=0}^{(n/2)} \frac{2^{n-2r+1}(-1)^r\, x^{n-2r}}{\sqrt{\pi}\, r\,!\,(n-2r)\,!} \tfrac{1}{2}\, \Gamma(n-r+\tfrac{1}{2})$$

$$\text{Since } 2 \int_0^\infty e^{-t^2} . t^{(2n-1)} dt = \Gamma(n)$$

$$= \sum_{r=0}^{(n/2)} \frac{2^{n-2r}\, (-1)^r\, x^{n-2r} . [2(n-r)]\,!}{\sqrt{\pi}\, r\,!\,(n-2r)\,!\, 2^{2(n-r)}(n-r)\,!} \sqrt{\pi}$$

$$\left(\text{Since } \Gamma(x+\tfrac{1}{2}) = \frac{(2x)\,!}{2^{2x}x\,!} \sqrt{\pi}\right)$$

$$= \sum_{r=0}^{(n/2)} (-1)^r \frac{(2n-2r)\,!}{2^n (r)\,!\,(n-2r)\,!\,(n-r)\,!} . x^{n-2r} = P_n(x).$$

Hence

$$P_n(x) = \frac{2}{\sqrt{\pi}\, n\,!} \int_0^\infty t^n e^{-t^2} H_n(xt)\, dt.$$

**Proved.**

**Ex. 6.** *Show that*

$$\sum_{r=0}^{n} \frac{H_k(x) H_k(z)}{2^k\, k\,!} = \frac{H_{n+1}(y) H_n(x) - H_{n+1}(x) H_n(y)}{2^{n+1}\, n\,!\, (y-x)}.$$

**Sol.** From recurrence formulae II, we have

$$x H_n(x) = n H_{n-1}(x) + \tfrac{1}{2} H_{n+1}(x) \qquad \text{...(i)}$$

$$\therefore \quad y H_n(y) = n H_{n-1}(y) + \tfrac{1}{2} H_{n+1}(y). \qquad \text{...(ii)}$$

Multiplying (ii) by $H_n(x)$ and (i) by $H_n(y)$ and than subtracting, we have

$$(y-x)\, H_n(x)\, H_n(y) = \tfrac{1}{2}\, [H_{n+1}(y)\, H_n(x) - H_{n+1}(x)\, H_n(y)]$$
$$- n[H_{n-1}(x)\, H_n(y) - H_{n-1}(y)\, H_n(x)]. \qquad \text{...(iii)}$$

Putting $n=0, 1, 2, 3, \ldots, (n-1) . n$ in (iii) respectively, we have

$$(y-x)\, H_0(x)\, H_0(y) = \tfrac{1}{2}[H_1(y) H_0(x) - H_1(x)\, H_0(y)] - 0 \qquad \text{...}(A_0)$$

$$(y-x)\,H_1(x)\,H_1(y)=\tfrac{1}{2}[H_2(y)\,H_1(x)-H_2(x)\,H_1(y)]$$
$$-1.[H_0(x)\,H_1(y)-H_0(y)\,H_1(x)] \quad \ldots (A_1)$$
$$(y-x)\,H_2(x)\,H_2(y)=\tfrac{1}{2}[H_3(y)\,H_2(x)-H_3(x)\,H_2(y)]$$
$$-2.[H_1(x)\,H_2(y)-H_1(y)\,H_2(x)] \quad \ldots (A_2)$$
$$(y-x)\,H_3(x)\,H_3(y)=\tfrac{1}{2}[H_4(y)\,H_3(x)-H_4(x)\,H_3(y)]$$
$$-3.[H_2(x)\,H_3(y)-H_2(y)\,H_3(x)] \quad \ldots (A_3)$$

... ... ... ... ... ... ... ... ... ... ...

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$$(y-x)\,H_{n-1}(x)\,H_{n-1}(y)=\tfrac{1}{2}\,[H_n(y)\,H_{n-1}(x)-H_n(x)\,H_{n-1}(y)]$$
$$-(n-1)\,[H_{n-2}(x)\,H_{n-1}(y)$$
$$-H_{n-2}(y)\,H_{n-1}(x)] \ldots (A_{n-1}).$$
$$(y-x)\,H_n(x)\,H_n(y)=\tfrac{1}{2}\,[H_{n+1}(y)\,H_n(x)-H_{n+1}(x)\,H_n(y)]$$
$$-n\,[H_{n-1}(x)\,H_n(y)$$
$$-H_{n-1}(y)\,H_n(x)] \ldots (A_n)$$

Multiplying $(A_0)$, $(A_1)$, $(A_2)$, $(A_3)$, $\ldots (A_{n-1})$, $(A_n)$ by

$$1,\ \frac{1}{2.1\,!},\ \frac{1}{2^2.2\,!},\ \frac{1}{2^3.3\,!}\ \ldots\ldots\ \frac{1}{2^{n-1}.(n-1)\,!},\ \frac{1}{2^n.n}\,;$$

respectively and adding, we have

$$(y-x)\sum_{k=0}^{n}\frac{H_k(x)\,H_k(y)}{2^k\,k\,!}=\frac{H_{n+1}(y)\,H_n(x)-H_{n+1}(x)\,H_n(y)}{2^{n+1}\,n\,!}$$

$$\therefore\quad \sum_{k=0}^{n}\frac{H_k(x)\,H_k(y)}{2^k\,k\,!}=\frac{H_{n+1}(y)\,H_n(x)-H_{n+1}(x)\,H_n(y)}{2^{n+1}\,n\,!\,(y-x)}$$

**Proved.**

## Exercise on Chapter VI

1. For what value of $n$, $H_n(0)=0$. [Meerut 78]

2. Prove that

$$H_n(x)=2^{n+1}.e^{x^2}\int_x^\infty e^{-t^2}\,t^{n+1}\,P_n\left(\frac{x}{t}\right)dt.$$ [B.H.U. 68]

[Hint. Proceed as in Ex. 5.]

3. Show that for $n=0, 1, 2, \ldots$

(i) $H_{2n}(x)=\dfrac{2^{n+1}(-1)^n}{\sqrt{\pi}}\,e^{x^2}\displaystyle\int_0^\infty e^{-t^2}.t^{2n}\cos 2xt\,dt.$ [Agra 77]

(ii) $H_{2n+1}(x)=\dfrac{2^{n+2}(-1)^n}{\sqrt{\pi}}\,e^{x^2}\displaystyle\int_0^\infty e^{-t^2}\,t^{2n+1}\sin 2xt\,dt.$

[Agra 81]

4. Show that $H_n^n(x)=2^n\,n\,!\,H_0(x)$

**5.** Show that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} \{H_n(x)\}^2 \, dx = \sqrt{\pi} 2^n \, n! \, (n+\tfrac{1}{2}).$$

[Meerut 82 (P)]

**6.** Show that $H_1(x) = 2x \, H_0(x)$. [Meerut 86]

**7.** Prove that $H_5(x) = 32x^5 - 160x^3 + 120x$. [Meerut 86]

[Hint. See § 6·6)

**8.** Show that

$$\sum_{n=0}^{\infty} \frac{H_{n+s}(x)}{n!} t^n = \exp(2xt - t^2) \, H_s(x-t).$$

[Jodhpur 83, 86]

**9.** If $\psi_n(x) = e^{-x^2/2} H_n(x)$, where $H_n(x)$ is a Hermites polynomial of degree $n$, then prove

$$I_{m,n} = \int_{-\infty}^{\infty} \psi_m(x) \, \psi_n(x) \, dx = 2^n \, n! \, \sqrt{\pi} \delta_{m,n}.$$

[Meerut 88]

# 7
# Laguerre Polynomials

## § 7·1. Laguerre's Differential Equation. [Meerut 78]

The differential equation

$$x \frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+\lambda y=0,$$

where $\lambda$ is a constant, is called Laguerre's differential equation.

## § 7·2. Solution of Laguerre's equation.

The Laguerre's equation is

$$x \frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+\lambda y=0 \quad \text{...(i)}$$

which may be solved by series integration.

Let us assume $y=\sum_{r=0}^{\infty} a_r x^{k+r}$ ...(ii)

as the solution of the given equation (i)

$$\therefore \quad \frac{dy}{dx}=\sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1}$$

and $$\frac{d^2y}{dx^2}=\sum_{n=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}.$$

Substituting in (1), we have

$$\sum_{r=0}^{\infty} a_r [(k+r)(k+r-1) x^{k+r-1}+(1-x)(k+r) x^{k+r-1}+\lambda x^{k+r}]=0$$

$$\sum_{r=0}^{\infty} a_r [(k+r)^2 x^{k+r-1}-(k+r-\lambda) x^{k+r}]=0. \quad \text{...(iii)}$$

Now (iii) being an identity, we can equate to zero the coefficient of various powers of $x$.

$\therefore$ Equating to zero the coefficient of lowest power of $x$ *i.e.* of $x^{k-1}$, we have $a_0 k^2=0.$

Now $a_0 \neq 0$, as it is the coefficient of the first term with which we start to write the series

$$\therefore \quad k=0.$$

Equating to zero the coefficient of general term, *i.e.* of $x^{k+r}$, we have $\quad a_{r+1}(k+r+1)^2 - a_r(k+r-\lambda) = 0$

$$\therefore \quad a_{k+1} = \frac{(k+r-\lambda)}{(k+r+1)^2} a_r$$

for $k=0$

$$a_{r+1} = \frac{r-\lambda}{(r+1)^2} a_r \qquad \text{...(iv)}$$

Putting $r=0, 1, 2, \ldots$, in (iv), we have

$$a_1 = \frac{-\lambda}{1} a_0 = (-1)\lambda a_0$$

$$a_2 = \frac{1-\lambda}{2^2} a_1 = (-1)^2 \frac{\lambda(\lambda-1)}{(2\,!)^2} a_0$$

$$a_3 = \frac{2-\lambda}{3^2} \cdot a_2 = (-1)^3 \frac{\lambda(\lambda-1)(\lambda-2)}{(3\,!)^2} a_0 \text{ etc.}$$

$$a_r = (-1)^r \frac{\lambda(\lambda-1)\ldots(\lambda-r+1)}{(r\,!)^2} a_0,$$

$\therefore$ From (ii), we have

$$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \ldots a_r x^r + \ldots$$

$$= a_0 \left[1 - \lambda x + \frac{\lambda(\lambda-1)}{(2\,!)^2} x^2 + \ldots + (-1)^r \frac{\lambda(\lambda-1)\ldots(\lambda-r+1)}{(r\,!)^2} x^r + \ldots\right] \qquad \text{...(v)}$$

If $\quad \lambda = n$

$$y = a_0 \left[1 - \frac{n}{1^2} \cdot x + \frac{n(n-1)}{(2\,!)^2} x^2 + \ldots + (-1)^r \frac{n(n-1)\ldots(n-r+1)}{(r\,!)^2} x^r + \ldots\right]$$

$$= a_0 \sum_{r=0}^{n} (-1)^r \frac{n(n-1)\ldots(n-r+1)}{(r\,!)^2} x^r$$

(The highest power of $x$ being $x^n$)

$$= a_0 \sum_{r=0}^{n} (-1)^r \frac{n\,!}{(n-r)\,!\,(r\,!)^2} \cdot x^r$$

## § 7·3. Laguerre Polynomials.

[Rohilkhand 83; 84; Meerut 73 (S)]

We define the standard solution of Laguerre equation

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+ny=0$$

as that for which $a_0=1$ and call it the Laguerre polynomial of order $n$, denote it by $L_n(x)$

$$\therefore\quad L_n(x)=\sum_{r=0}^{n}(-1)^r\frac{n\,!}{(n-r)\,!\,(r\,!)^2}\,.x^r.$$

**Note.** Some authors define $L_n(x)$ by taking $a_0=n\,!$ *i.e.*

$$L_n(x)=\sum_{r=0}^{n}(-1)^r\frac{(n\,!)^2}{(n-r)\,!\,(r\,!)^2}\,x^r.$$

## § 7·4. Generating Function.

**To prove** $\dfrac{1}{(1-t)}\,.e^{-tx/(1-t)}=\sum_{n=0}^{\infty}t^n\,L_n(x).$

[Meerut 82 (P); Rohilkhand 80, 85]

**Proof.** We have

$$\frac{1}{(1-t)}\,e^{-tx/(1-t)}=\frac{1}{1-t}\sum_{r=0}^{\infty}\frac{1}{r\,!}\left(-\frac{xt}{1-t}\right)^r$$

$$=\sum_{r=0}^{\infty}.\frac{(-1)^r}{r\,!}\,\frac{x^r.t^r}{(1-t)^{r+1}}$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!}\,.x^rt^r.(1-t)^{-(r+1)}$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!}\,x^r\,t^r\left[1+(r+1)\,t+\frac{(r+1)\,(r+2)}{2\,!}\,t^2+\ldots\right]$$

$$=\sum_{r=0}^{\infty}\left[\frac{(-1)^r}{r\,!}\,x^r\,t^r\sum_{s=0}^{\infty}\frac{(r+s)\,!}{r\,!\,s\,!}\,.t^s\right]$$

$$=\sum_{r,s=0}^{\infty}(-1)^r\frac{(r+s)\,!}{(r\,!)^2\,s\,!}\,x^r\,t^{r+s}.$$

Coefficient of $t^n$, for a fixed value of $r$ is given by

$$(-1)^r\frac{n\,!}{(r\,!)^2\,(n-r)\,!}\,.x^r$$

obtained by putting $s+r=n$ $\qquad\therefore\quad s=n-r$

Therefore the total coefficient of $t^n$ is obtained by summing over all allowed values of $r$.

Since $s=n-r$ and $s \geqslant 0$.

$\therefore \quad n-r \geqslant 0$ or $r \leqslant n$.

Hence the coefficient of $t^n$ is

$$\sum_{r=0}^{\infty} (-1)^r \frac{n\,!}{(r\,!)^2\,(n-r)\,!}\, x^r = L_n(x).$$

Hence $\dfrac{1}{(1-t)}\, e^{-tx/(1-t)} = \sum_{n=0}^{\infty} t^n\, L_n(x)$. **Proved.**

**Note.** If we take $L_n(x) = \sum_{r=0}^{\infty} (-1)^r \dfrac{(n\,!)^2}{(n-r)\,!\,(r\,!)^2} \cdot x^r$

then $\dfrac{1}{(1-t)}\, e^{-tx/(1-t)} = \sum_{n=0}^{\infty} \dfrac{t^n}{n\,!}\, L_n(x)$. **[Meerut 82, 84; Agra 76]**

**§ 7·5. Other forms for the Laguerre polynomials. (Rodrigues formula)**

**To prove** $\mathbf{L_n(x) = \dfrac{e^x}{n\,!}\dfrac{d^n}{dx^n}\,(x^n e^{-x})}$

**[Kanpur 81, 82; Meerut 76, 77(S), 81, 83 (P), 85, 89; Rohilkhand 80, 89; Raj. 79, 83, 85]**

**Proof.** $\dfrac{e^x}{n\,!}\dfrac{d^n}{dx^n}\,(x^n e^{-x}) = \dfrac{e^x}{n\,!}\Big[x^n(-1)^n\, e^{-x} + n.nx^{n-1}.(-1)^{n-1}e^{-x}$

$+\dfrac{n\,(n-1)}{2}.n(n-1)\,x^{n-2}\,(-1)^{n-2}e^{-x} + \ldots + n\,!.e^{-x}\Big]$

(Using Leibnitz's theorem)

$= \dfrac{e^x}{n\,!}.e^{-x}\left[(-1)^n\, x^n + (-1)^{n-1}\,\dfrac{n.n\,!}{(n-1)!}\, x^{n-1} + \ldots + n\,!\right]$

$= (-1)^n\,\dfrac{n\,!}{(n\,!)^2}.x^n + (-1)^{n-1}\,\dfrac{n\,!}{\{(n-1)!\}^2.1!}\, x^{n-1} + \ldots + \dfrac{n\,!}{n\,!}$

$$= \sum_{r=0}^{\infty} (-1)^r\,\frac{n\,!}{(r\,!)^2\,(n-r)\,!}\, x^r = L_n(x).$$

**Proved.**

Hence $L_n(x) = \dfrac{e^x}{n\,!}\dfrac{d^n}{dx^n}\,(x^n.e^{-x})$.

**Note.** If we make $L_n(x) = \sum_{r=0}^{\infty} (-1)^r\,\dfrac{(n\,!)^2}{(r\,!)^2\,(n-r)\,!}\, x^r$

then $\quad L_n(x)=e^x \dfrac{d^n}{dx^n}(x^n e^{-x}).$

[Raj. 77; Meerut 79, 83, 84 (P);
Agra 85, 87]

**§ 7·6. To find first few Laguerre polynomials.**

$$L_n(x)=\frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}).$$

Putting $n=0, 1, 2, 3, \ldots,$ etc.

$$L_0(x)=\frac{e^x}{0!}(x^0 e^{-x})=1$$

$$L_1(x)=\frac{e^x}{1!}(xe^{-x})=e^x(e^{-x}-xe^{-x})=1-x$$

$$L_2(x)=\frac{e^x}{2!}\frac{d^2}{dx^2}(x^2e^{-x})=\frac{e^x}{2!}\frac{d}{dx}(2xe^{-x}-x^2e^{-x})$$

$$=\frac{e^x}{2!}(2e^{-x}-4xe^{-x}+x^2e^{-x})=\frac{1}{2!}(2-4x+x^2)$$

Similarly,

$$L_3(x)=\frac{1}{3!}(6-18x+9x^2-x^3). \qquad \text{[Meerut 81 (P)}$$

$$L_4(x)=\frac{1}{4!}(24-96x+72x^2-16x^3+x^4) \text{ etc.}$$

Laguerre's polynomials may also be obtained directly.

[from § 7·3]

**Note.** If we take $L_n(x)=e^x \dfrac{d^n}{dx^n}(x^n e^{-x}),$

then
$$L_0(x)=1,\ L_1(x)=1-x$$
$$L_2(x)=2-4x-x^2$$
$$L_3(x)=6-18x+9x^2-x^3$$
$$L_4(x)=24-96x+72x^2-16x^3+x^4 \text{ etc.}$$

**§ 7·7. Orthogonal property of the Laguerre polynomials.**

**To prove** $\displaystyle\int_0^\infty e^{-x} L_n(x)\, L_m(x)\, dx=\delta_{mn}=\begin{cases}0 & \text{if } m\neq n\\ 1 & \text{if } m=n\end{cases}$

[Meerut 78, 80(S), 81, 86(R), 87; Agra 69, 82; Raj. 80, 86;
Rohilkhand 82, 86]

**Proof.** We have

$$\sum_{n=0}^{\infty} t^n L_n(x)=\frac{1}{1-t}e^{-tx/(1-t)}$$

and $\displaystyle\sum_{m=0}^{\infty} s^m L_m(x)=\frac{1}{1-s}e^{-sx/(1-s)}$

$$\therefore \sum_{m,n=0}^{\infty} e^{-x}\, t^n\, s^m\, L_n(x)\, L_m(x)$$

$$= e^{-x}.\ \frac{1}{(1-t)\,(1-s)}\, e^{-xt/(1-t)}\, e^{-sx/(1-s)}$$

Thus

$$\int_0^\infty e^{-x}\, L_n(x)\, L_m(x)\, dx$$

= coeffi. of $t^n\, s^m$ in the expansion of

$$\int_0^\infty e^{-x}.\frac{1}{(1-t)\,(1-s)}\,.e^{-tx/(1-t)}.e^{-sx/(1-s)}\, dx$$

Now $\displaystyle\int_0^\infty e^{-x}.\frac{1}{(1-t)\,(1-s)}\ e^{-tx/(1-t)}.e^{-sx/(1-s)}\, dx$

$$=\frac{1}{(1-t)}\,\frac{1}{(1-s)}\int_0^\infty e^{-x[1+t/(1-t)+s/(1-s)]}\, dx$$

$$=\frac{1}{(1-t)\,(1-s)}\cdot\frac{1}{1+t/(1-t)+s/(1-s)}$$

$$\left[e^{-x\,[1+t/(1-t)+s/(1-s)]}\right]_0^\infty$$

$$=-\frac{1}{(1-t)\,(1-s)}\cdot\frac{(1-t)\,(1-s)}{(1-t)\,(1-s)+t\,(1-s)+s\,(1-t)}\ [0-1]$$

$$\therefore \frac{1}{1-st}=(1-ts)^{-1}=1+ts+(ts)^2+\ldots+(ts)^n+\ldots$$

in which coefficient of $t^n\, s^m$

is 0 if $m\neq n$

and is 1 if $m=n$.

Hence $\displaystyle\int_0^\infty e^{-x}\, L_n(x)\, L_m(x)\, dx=\begin{cases}0 & \text{if } m\neq n\\ 1 & \text{if } m=n\end{cases}$

or $\displaystyle\int_0^\infty e^{-x}\, L_n(x)\, L_m(x)\, dx=\delta_{mn}.$

Note. $\displaystyle\int_0^\infty e^{-x}\, L_n(x)\, L_m(x)\, dx=n\,!\ m\,!\ \delta_{mn}$ [Meerut 87]

if we take $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n\,!}\, t^n\, L_n(x)=\frac{1}{1-t}\, e^{-tx/(1-t)}.$

[Agra 77; Raj. 78; Meerut 84 (P), 86]

**§ 7·8. Recurrence formulae for Lagurre Polynomials.**

**(I) $(n+1)\, L_{n+1}(x)=(2n+1-x)\, nL_n(x)-L_{n-1}(x)$.**

**[Raj. 86; Meerut 72, 75, 80, 81, 83 (P), 85, 90; Rohilkhand 84]**

**Proof.** We have $\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} \cdot e^{-tx/(1-t)}$

Differentiating both sides w.r.t. '$t$', we have

$$\sum_{n=1}^{\infty} nt^{n-1} L_n(x) = \frac{1}{(1-t)^2} \cdot e^{-tx/(1-t)} + \frac{1}{1-t} e^{-tx/(1-t)} \left\{ \frac{-x}{(1-t)^2} \right\}$$

or $(1-t)^2 \sum_{n=1}^{\infty} nt^{n-1} L_n(x) = (1-t) \frac{e^{-tx/(1-t)}}{(1-t)} - x \cdot \frac{1}{1-t} e^{-tx/(1-t)}$

or $(1-t)^2 \sum_{n=1}^{\infty} nt^{n-1} L_n(x) = (1-t) \sum_{n=0}^{\infty} t^n L_n(x) - x \sum_{n=0}^{\infty} t^n L_n(x)$

or $(1-2t+t^2) \sum_{n=1}^{\infty} nt^{n-1} L_n(x) = (1-t) \sum_{n=0}^{\infty} t^n L_n(x) - x \sum_{n=0}^{\infty} t^n L_n(x)$

or $\sum_{n=1}^{\infty} nt^{n-1} L_n(x) - 2 \sum_{n=1}^{\infty} nt^n L_n(x) + \sum_{n=1}^{\infty} nt^{n+1} L_n(x)$

$$= \sum_{n=0}^{\infty} t^n L_n(x) - \sum_{n=0}^{\infty} t^{n+1} L_n(x) - x \sum_{n=0}^{\infty} t^n L_n(x)$$

$\therefore$ Equating the coefficient of $t^n$ on the two sides, we have

$$(n+1) L_{n+1}(x) - 2n L_n(x) + (n-1) L_{n-1}(x) = L_n(x) - L_{n-1}(x) - xL_n(x)$$

or $(\mathbf{n+1}) \mathbf{L_{n+1}(x) = (2n+1-x) L_n(x) - nL_{n-1}(x)}$. **Proved.**

**Note.** If we take $\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) = \frac{1}{1-t} e^{-tx/(1-t)}$

then proceeding similarly, we have

$$L_{n+1}(x) = (2n+1-x) L_n(x) - n^2 L_{n-1}(x)$$

**[Agra 77, 80, 83; Meerut 74, 74 (S), 87; Agra 87]**

**(II)** $\mathbf{xL'_n(x) = nL_n(x) - nL_{n-1}(x)}$.

**[Agra 76; B.H.U. 78; Meerut 78, 83 (P), 86 (R); Raj. 85]**

**Proof.** We have $\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} e^{-tx/(1-t)}$

Differentiating both sides, w.r.t. '$x$', we have

$$\sum_{n=0}^{\infty} t^n L'_n(x) = \frac{1}{(1-t)} e^{-tx/(1-t)} \cdot \left( \frac{-t}{1-t} \right)$$

or $(1-t)\sum_{n=0}^{\infty} t^n L'_n(x)=-t.\frac{1}{1-t}\cdot e^{-tx/(1-t)}$

or $$(1-t)\sum_{n=0}^{\infty} t^n L'_n(x)=-t\sum_{n=0}^{\infty} t^n L_n(x)$$

or $$\sum_{n=0}^{\infty} t^n L'_n(x)-\sum_{n=0}^{\infty} t^{n+1} L'_n(x)=-\sum_{n=0}^{\infty} t^{n+1} L_n(x).$$

Equating the coefficients of $t^n$, on both the sides, we have

$$L'_n(x)-L'_{n-1}(x)=-L_{n-1}(x)$$

or $$\mathbf{L'_n(x)=L'_{n-1}(x)-L_{n-1}(x)} \qquad \text{...(i)}$$

Differentiating recurrence formula I w.r.t. '$x$', we have

$$(n+1)\, L'_{n+1}(x)=(2n+1-x)\, L'_n(x)-L_n(x)-nL'_{n-1}(x). \qquad \text{...(ii)}$$

Replacing $n$ by $(n+1)$ in (i), we have

$$L'_{n+1}(x)=L'_n(x)-L_n(x).$$

Also from (i) $L'_{n-1}(x)=L'_n(x)+L_{n-1}(x)$.

Substituting these values in (ii), we have

$$(n+1)\,\{L'_n(x)-L_n(x)\}$$
$$=(2n+1-x)\,L'_n(x)-L_n(x)-n\,\{L'_n(x)+L_{n-1}(x)\}$$

or $$\mathbf{xL'_n(x)=nL_n(x)-nL_{n-1}(x).}$$ **Proved.**

**(III)** $\mathbf{L'_n(x)=-\sum_{r=0}^{n-1} L_r(x).}$ **[Meerut 81 (P)]**

**Proof.** We have $\sum_{n=0}^{\infty} t^n L_n(x)=\frac{1}{1-t}\, e^{-tx/(1-t)}$

Differentiating w.r.t. '$x$', we have

$$\sum_{n=0}^{\infty} t^n L'_n(x)=-\frac{t}{1-t}\sum_{r=0}^{\infty} t^r L_r(x) \qquad \text{(as in II)}$$

$$=-t.(1-t)^{-1}\sum_{r=0}^{\infty} t^r L_r(x)$$

$$=-t\,(1+t+t^2+\dots)\sum_{r=0}^{\infty} t^r L_r(x)$$

$$=-t\sum_{s=0}^{\infty} t^s.\sum_{r=0}^{\infty} t^r L_r(x)=-\sum_{s=0,r=0}^{\infty} t^{r+s+1} L_r(x). \qquad \text{...(i)}$$

For fixed value of $r$, the coefficient of $t^n$ on the R.H.S. is $-L_r(x)$.

obtained by putting $r+s+1=n$

or $s=n-r-1$

Total coefficient of $t^n$ is obtained by summing over all allowed values of $r$.

Since $s=n-r-1$ and $s \geqslant 0$.

Therefore $n-r-1 \geqslant 0$ or $r \leqslant (n-1)$.

$\therefore$ Coefficient of $t^n$ on the R.H.S. $= -\sum_{r=0}^{n-1} L_r(x)$.

Therefore equating coefficient of $t^n$, on the two sides of (i),

we have $$L'_n(x) = -\sum_{r=0}^{\infty} L_r(x).$$ **Proved.**

## EXAMPLES

**Ex. 1.** *Prove that* $L_n(0)=1$. **[Meerut 78, 86 (R); Agra 86]**

**Sol.** We have $\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} e^{-tx/(1-t)}$

Putting $x=0$, we have

$$\sum_{n=0}^{\infty} t^n L_n(0) = \frac{1}{1-t} = (1-t)^{-1}$$

$$= 1+t+t^2+\ldots+t^n+\ldots$$

$$= \sum_{n=0}^{\infty} t^n$$

$\therefore$ $L_n(0)=1$. **Proved.**

**Ex. 2.** *Prove that*

$$xL''_n(x)+(1-x)\,L'_n(x)+nL_n(x)=0.$$

**[Agra 78, Meerut 83 (P)]**

*and hence deduce that*

$$L'_n(0)=-n.$$ **[Meerut 78]**

**Sol.** Since $L_n(x)$ satisfies the Laguerre's equation

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+ny=0 \quad \text{for } \lambda=n$$

$\therefore$ $xL''_n(x)+(1-x)\,L'_n(x)+nL_n(x)=0$. **Proved.**

Putting $x=0$, we have

$$L'_n(0)=-nL_n(0)$$

or $$L'_n(0)=-n.$$ **Proved.**

(Since $L_n(0)=1$, see ex. 1.)

**Associated Laguerre Polynomials.**

**§ 7·9. Associated Laguerre Equation** **[Kanpur 83]**

*The differential equation*

$$x\frac{d^2y}{dx^2}+(\alpha+1-x)\frac{dy}{dx}+ny=0$$

*is called* **associated Laguerre equation.**

**§ 7·10. If v is a solution of Laguerre equation of order n+α then $\frac{d^\alpha v}{dx^\alpha}$ satisfies Laguerre's associated equation.**

**Proof.** Since $v$ is a solution of Laguerre's equation of order $(n+\alpha)$

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+(n+\alpha)\,y=0$$

$$x\frac{d^2v}{dx^2}+(1-x)\frac{dv}{dx}+(n+\alpha)\,v=0.$$

Differentiating (i) $\alpha$ times w.r.t. '$x$' by Leibnitz's theorem, we have

$$\left(x\frac{d^{\alpha+2}v}{dx^{\alpha+2}}+\alpha.1.\frac{d^{\alpha+1}v}{dx^{\alpha+1}}\right)+\left((1-x)\frac{d^{\alpha+1}v}{dx^{\alpha+1}}\,(-1).\alpha\,\frac{d^\alpha v}{dx^\alpha}\right)$$

$$+(n+\alpha)\frac{d^\alpha v}{dx^\alpha}=0$$

or $$\frac{d^{\alpha+2}v}{dx^{\alpha+2}}+(\alpha+1-x)\frac{d^{\alpha+1}v}{dx^{\alpha+1}}+n\frac{d^\alpha v}{dx^\alpha}=0$$

which shows that

$$y=\frac{d^\alpha v}{dx^\alpha}$$

satisfies Laguerre's associated equation

$$x\frac{d^2y}{dx^2}+(\alpha+1-x)\frac{dy}{dx}+ny=0.$$ **Proved.**

**§ 7·11. Associated Laguerre's Polynomials.** **[Meerut 85]**

From § 7·10 we see that if $v$ is the solution of Laguerre's equation of order $(n+\alpha)$ *i.e.* $v=L_{n+\alpha}(x)$ (Laguerre's polynomial of order $n+\alpha$), then $\frac{d^\alpha v}{dx^\alpha}=\frac{d^\alpha}{dx^\alpha}L_{n+\alpha}(x)$ satisfies the associated Laguerre equation.

Thus we define the associated Laguerre polynomial by

$$L^\alpha_n(x)=(-1)^\alpha\frac{d^\alpha}{dx^\alpha}L_{n+\alpha}(x).$$

**§ 7·12. Prove that**

$$\mathbf{L}_n^\alpha(\mathbf{x}) = \sum_{r=0}^{n} (-1)^r \frac{(n+\alpha)!}{(n-r)!\,(\alpha+r)!\,(r!)}\,\mathbf{x}^r$$

[Meerut 76; Kanpur 83]

**Proof.** Since $L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!\,(r!)^2} . x^r$

$$\therefore \quad L_{n+\alpha}(x) = \sum_{r=0}^{n+\alpha} (-1)^r \frac{(n+\alpha)!}{(n+\alpha-r)!\,(r!)^2} . x^r.$$

Differentiating $\alpha$-times, w.r.t. '$x$', we have

$$\frac{d^\alpha}{dx^\alpha} L_{n+\alpha}(x) = \frac{d^\alpha}{dx^\alpha} \sum_{r=0}^{n+\alpha} (-1)^r \frac{(n+\alpha)!}{(n+\alpha-r)!\,(r!)^2} . x^r$$

$$= \frac{d^\alpha}{dx^\alpha} \sum_{r=0}^{n+\alpha} (-1)^r \frac{(n+\alpha)!}{(n+\alpha-r)!\,(r!)^2} . x^r$$

$$\left(\text{Since } \frac{d^\alpha}{dx^\alpha} x^r = 0 \text{ where } 0 \leqslant r < \alpha\right)$$

$$= \sum_{r=\alpha}^{n+\alpha} (-1)^r \frac{(n+\alpha)!}{(n+\alpha-r)!\,(r!)^2} . r(r-1) \ldots (r-\alpha+1)\, x^{r-\alpha}$$

$$= \sum_{r=\alpha}^{n+\alpha} (-1)^r \frac{(n+\alpha)!}{(n+\alpha-r)!\,(r!)^2} \frac{r!}{(r-\alpha)!} . x^{r-\alpha}$$

$$= \sum_{s=0}^{n} (-1)^{\alpha+s} \frac{(n+\alpha)!}{(n-s)\,(\alpha+s)!} \frac{1}{(s)!} x^s$$

Taking $r-\alpha=s$

$$= (-1)^\alpha \sum_{r=0}^{n} (-1)^r . \frac{(n+\alpha)!}{(n-r)!\,(\alpha+r)!\,r!} x^r$$

$$\therefore \quad L^\alpha{}_n(x) = (-1)^\alpha . \frac{d^\alpha}{dx^\alpha} L_{n+\alpha}(x)$$

$$= \sum_{r=0}^{n} (-1)^r \frac{(n+\alpha)!}{(n-r)!\,(\alpha+r\;!\,r!} . x^r.$$

**Proved.**

## § 7·13. Generating Function.

To prove $\dfrac{1}{(1-t)^{\alpha+1}} e^{-xt/(1-t)} = \sum\limits_{n=0}^{\infty} t^{\alpha} L_n^{\alpha}(x) \cdot t^n$. [Kanpur 86]

**Proof.** If $L_n(x)$ is the Laguerre polynomial of degree $n$ from § 7 4, we have

$$\frac{1}{(1-t)} e^{-xt/(1-t)} = \sum_{n=0}^{\infty} t^n . L_n(x).$$

Differentiating both sides $\alpha$ times w.r.t. '$x$' taking $t$ as constant, we have

$$\frac{1}{(1-t)} \cdot \frac{d^{\alpha}}{dx^{\alpha}} e^{-xt/(1-t)} = \frac{d^{\alpha}}{dx^{\alpha}} \sum_{n=0}^{\infty} t^n . L_n(x).$$

or $$\frac{1}{(1-t)} \left\{\frac{-t}{(1-t)}\right\}^{\alpha} e^{-xt/(1-t)}$$

$$= \frac{d^{\alpha}}{dx^{\alpha}} \sum_{n=0}^{\infty} t^{n+\alpha} L_{n+\alpha}(x)$$

$$\left(\text{Since } \frac{d^{\alpha}}{dx^{\alpha}} L_n(x) = 0 \text{ where } 0 \leqslant n < \alpha \right.$$

$$\left. \text{as } L_n(x) \text{ is a polynomial of order } n \right)$$

or $$(-1)^{\alpha} \frac{t^{\alpha}}{(1-t)^{\alpha+1}} e^{-xt/(1-t)} = \sum_{n=0}^{\infty} t^{n+\alpha} \frac{d^{\alpha}}{dx^{\alpha}} L_{n+\alpha}(x)$$

or $$(-1)^{2\alpha} \cdot \frac{1}{(1-t)^{\alpha+1}} e^{-xt/(1-t)} = \sum_{n=0}^{\infty} t^n (-1)^{\alpha} \frac{d^{\alpha}}{dx^{\alpha}} L_{n+\alpha}(x)$$

or $$\frac{1}{(1-t)^{\alpha+1}} e^{-xt/(1-t)} = \sum_{n=0}^{\infty} t^n L^{\alpha}_n(x).$$ **Proved.**

## § 7·14. Other form for associated Laguerre Polynomial.

[Rodrigues formula]

**To prove**

$$L^{\alpha}_n(x) = \frac{e^x x^{-\alpha}}{n!} \cdot \frac{d^n}{dx^{\alpha}} (e^{-x} . x^{n+\alpha}).$$

[Raj. 84. 86; Jodhpur 84; Meerut 71, 77, 85; Kanpur 81, 82]

**Proof.** We have

$$\frac{e^x . x^{-\alpha}}{n!} \cdot \frac{d^n}{dx^n} (e^{-x} . x^{n+\alpha})$$

$$=\frac{e^{x}\cdot x^{-\alpha}}{n!}\Big[x^{n+\alpha}.(-1)^n\, e^{-x}+n.(n+\alpha)\, x^{n+\alpha-1}\, e^{-x}$$

$$+\frac{n\,(n-1)}{2\,!}.(n+\alpha)\,(n+\alpha-1)\, x^{n+\alpha-2}\,(-1)^{n-2}\, e^{-x}+\ldots$$

$$+n\,(-1)\, e^{-x}.\frac{(n+\alpha)\,!}{(\alpha+1)\,!}.x^{\alpha+1}+e^{-x}.\frac{(n+\alpha)\,!}{\alpha\,!}.x^{\alpha}\Big]$$

[Using Leibnitz's theorem]

$$=\frac{(-1)^n\, x^n}{n\,!}+(-1)^{n-1}.\frac{(n+\alpha)}{(n-1)}\, x^{n-1}$$

$$+(-1)^{n-2}.\frac{(n+\alpha)\,(n+\alpha-1)}{2\,!\cdot(n-2)\,!}\, x^{n-2}$$

$$+\ldots+(-1)\frac{(n+\alpha)\,!}{(\alpha+1)\,!\,(n-1)\,!}+\frac{(n+\alpha)\,!}{\alpha\,!\,n\,!}$$

$$=(-1)^n\cdot\frac{(n+\alpha)\,!}{(n-n)\,!\,(\alpha+n)\,!\,n\,!}.x^n$$

$$+(-1)^{n-1}.\frac{(n+\alpha)\,!}{\{n-(n-1)\,!\,(\alpha+(n-1)\}\,!\,(n-1)\,!}.x^{n-1}$$

$$+\ldots+(-1)\frac{(n+\alpha)\,!}{(n+1)\,!\,(\alpha+1)\,!\,1\,!}\cdot x$$

$$+\frac{(n+\alpha)\,!}{n\,!.\alpha\,!.0\,!}$$

$$=\sum_{n=0}^{n}(-1)^r\,\frac{(n+\alpha)\,!}{(n-r)\,!\,(\alpha+r)\,!\,r\,!}.\,x^r$$

$$=L^{\alpha}{}_{n}(x).$$ **Proved.**

## § 7·15. Orthogonal property of the associated Laguerre Polynomials.

**To prove**

$$\int_0^{\infty} e^{-x}\, x^{\alpha}\, L^{\alpha}{}_{n}(x)\, L^{\alpha}{}_{m}(x)\, dx=\frac{(n+\alpha)\,!}{n\,!}\,\delta_{mn}.$$

**Proof.** We have

$$\sum_{n=0}^{\infty} t^n\, L^{\alpha}{}_{n}(x)=\frac{1}{(1-t)^{\alpha+1}}\, e^{-xt/(1-t)}$$

and
$$\sum_{m=0}^{\infty} s^m\, L^{\alpha}{}_{m}(x)=\frac{1}{(1-s)^{\alpha+1}}\, e^{-xs/(1-s)}$$

$$\therefore\quad \sum_{m,\,n=0}^{\infty} t^n s^m e^{-x}.x^{\alpha}\, L_n^{\alpha}\,(x)\, L_m^{\alpha}\,(x)=\frac{e^{-\alpha}\, x^{\alpha}}{(1-t)^{\alpha+1}\,(1-s)^{\alpha+1}}.e^{-tx/(1-t)}$$
$$\times e^{-xs/(1-s)}$$

$$=\frac{x^{\alpha}}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}.e^{-x.[1+t/(1-t)+s/(1-s)]}$$

$$=\frac{x^{\alpha}}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}.e^{-x.(1-ts)/(1-t)\,(1-s)}$$

$$\therefore \quad \int_0^{\infty} e^{-x}\,x^{\alpha}\,L_n^{\alpha}(x)\,L_m^{\alpha}(x)\,dx$$

=Coeff. of $t^n s^m$ in the expansion of

$$\int_0^{\infty}\frac{x^{\alpha}}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}.e^{-x\,(1-ts)/(1-t)\,(1-s)}\,dx$$

$$\text{Now} \int_0^{\infty}\frac{x^{\alpha}}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}.e^{\frac{-(1-ts)\,x}{(1-t)\,(1-s)}}\,dx$$

$$=\frac{1}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}.\int_0^{\infty}x^{\alpha}.e^{\frac{-(1-ts)\,x}{(1-t)\,(1-s)}}\,dx$$

Integrating by parts taking $x^{\alpha}$ as the first function

$$=\frac{1}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}\left[\left\{-x^{\alpha}.\frac{(1-t)\,(1-s)}{(1-ts)}\,.\,e^{\frac{-(1-ts)\,x}{(1-t)\,(1-s)}}\right\}_0^{\infty}\right.$$

$$\left.+\frac{\alpha\,(1-t)\,(1-s)}{(1-ts)}\int_0^{\infty}x^{\alpha-1}.e^{\frac{-(1-ts)\,x}{(1-t)\,(1-s)}}\,dx\right]$$

$$=\frac{1}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}\;\frac{\alpha\,(1-t)\,(1-s)}{(1-ts)}$$

$$\times\int_0^{\infty}x^{\alpha-1}\,.\,e^{\frac{-(1-ts)\,x}{(1-t)\,(1-s)}}\,dx$$

proceeding similarly $(\alpha-1)$ times, we have

$$=\frac{1}{(1-t)^{\alpha+1}(1-s)^{\alpha+1}}\;\frac{\alpha\,(\alpha-1)\ldots2.\,1.(1-t)^{\alpha}\,(1-s)^{\alpha}}{(1-ts)^{\alpha}}$$

$$\times\int_0^{\infty}e^{\frac{-(1-ts)\,x}{(1-t)\,(1-s)}}\,dx$$

$$=\frac{\alpha\,!}{(1-t)\,(1-s)\,(1-ts)^{\alpha}}\left[-\frac{(1-t)\,(1-s)}{(1-ts)^{\alpha}}\,.e^{\frac{-(1-ts)\,x}{(1-t)\,(1-s)}}\right]_0^{\infty}$$

$$=\frac{\alpha\,!}{(1-ts)^{\alpha+1}}$$

$$=\alpha\,!.(1-ts)^{-(\alpha+1)}$$

$$=\alpha\,!\left[1+(\alpha+1)\,ts+\frac{(\alpha+1)(\alpha+2)}{2\,!}\,t^2s^2+\ldots\right.$$

$$\left.+\frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n\,!}\,(ts)^n+\ldots\right]$$

in which coefficient of $t^n s^m$ is 0 if $m \neq n$ and is

$$\frac{\alpha\,!\,(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n\,!}=\frac{(n+\alpha)\,!}{n\,!}\text{ if } m=n.$$

Hence $\displaystyle\int_0^\infty e^{-x}\,x^\alpha L_n^\alpha(x)\,L_m^\alpha(x)\,dx=\frac{(n+\alpha)\,!}{n\,!}\,\delta_{mn}$ **Proved.**

**§ 7 16. Recurrence formulae for associated Laguerre polynomials.**

(1) $L^\alpha_{n-1}(x)+L_n^{\alpha-1}(x)=L_n^\alpha(x)$. **[Kanpur 71, 83, 87 ; Agra 84 ; Raj. 85; Jodhpur 83]**

**Proof.** We have

$$L_n^\alpha(x)=\sum_{r=0}^{n}(-1)^r\frac{(n+\alpha)\,!}{(n-r)\,!\,(\alpha+r)\,!\,r\,!}\,x^r$$

$$\therefore\ L^\alpha_{n-1}(x)+L_n^{\alpha-1}(x)=\sum_{r=0}^{n-1}(-1)^r\frac{(n-1+\alpha)\,!}{(n-1+r)\,!\,(\alpha+r)\,!\,r\,!}\,x^r$$

$$+\sum_{r=0}^{n}(-1)^r\frac{(n+\alpha-1)\,!\,x^r}{(n-r)\,!\,(\alpha-1+r)\,!\,r\,!}$$

$$=\sum_{r=0}^{n-1}\frac{(-1)^r}{r\,!}\cdot\left[\frac{(n-1+\alpha)\,!}{(n-1-r)\,!\,(\alpha+r)\,!}+\frac{(n-1+\alpha)\,!}{(n-r)\,!\,(\alpha-1+r)\,!}\right]x^r$$

$$+(-1)^n\cdot\frac{x^n}{n\,!}$$

$$=\sum_{r=0}^{n-1}(-1)^r\frac{(n-1+\alpha)\,!}{(n-1-r)\,!\,(\alpha+r-1)\,!\,r!}$$

$$\times\left[\frac{1}{\alpha+r}+\frac{1}{n-r}\right]x^r+(-1)^n\frac{x^n}{n\,!}$$

$$=\sum_{r=0}^{n-1}(-1)^r\frac{(n-1+\alpha)\,!}{(n-r-1)\,!\,(\alpha+r-1)\,!\,r\,!}$$

$$\times\frac{(n+\alpha)}{(\alpha+r)(n-r)}\,x^r+(-1)^n\frac{x^n}{n\,!}$$

$$=\sum_{r=0}^{n-1}(-1)^r\frac{(n+\alpha)\,!}{(n-r)\,!\,(\alpha+r)\,!\,r\,!}\,x^r+(-1)^n\frac{x^n}{n}$$

$$= \sum_{r=0}^{n} (-1)^r \frac{(n+\alpha)\,!}{(n-r)\,!\,(\alpha+r)\,!\,r\,!}\, x^r = L^\alpha_n(x)$$

or $$L^\alpha_{n-1}(x)+L_n^{\alpha-1}(x)=L^\alpha_n(x).$$ **Proved.**

(II) $(n+1)\, L^\alpha_{n+1}(x)=(2n+\alpha+1-x)\, L^\alpha_n(x)-(n+\alpha)\, L^\alpha_{n-1}(x)$

**Proof.** From recurrence formula (1) for Laguerre polynomials, we have

$$(n+1)\, L_{n+1}(x)=(2n+1-x)\, L_n(x)-nL_{n-1}(x).$$

Replacing $n$ by $(n+\alpha)$, we have

$$(n+\alpha+1)\, L_{n+\alpha+1}(x)=(2n+2\alpha+1-x)\, L_{n+\alpha}(x) -(n+\alpha)\, L_{n+\alpha-1}(x)$$

Differentiating both sides $\alpha$ times w.r.t. '$x$', we have

$$(n+\alpha+1)\frac{d^\alpha}{dx^\alpha} L_{n+\alpha+1}(x)=(2n+2\alpha+1-x)\frac{d^\alpha}{dx^\alpha} L_{n+\alpha}(x) +\alpha(-1)\frac{d^{\alpha-1}}{dx^{\alpha-1}} L_{n+\alpha}(x)-(n+\alpha)\frac{d^\alpha}{dx^\alpha} L_{n+\alpha-1}(x)$$

or $$(n+\alpha+1)(-1)^\alpha \frac{d^\alpha}{dx^\alpha} L_{n+\alpha+1}(x)=(2n+2\alpha+1)(-1)^\alpha \frac{d^\alpha}{dx^\alpha} L_{n+\alpha}(x) -x(-1)^\alpha \frac{d^\alpha}{dx^\alpha} L_{n+\alpha}(x)+\alpha(-1)^{\alpha-1}\frac{d^{\alpha-1}}{dx^{\alpha-1}} L_{n+\alpha-1}(x) -(n+\alpha)(-1)^\alpha \frac{d^\alpha}{dx^\alpha} L_{n+\alpha-1}(x)$$

or $$(n+\alpha+1)\, L^\alpha_{n+1}(x)=(2n+2\alpha+1)\, L^\alpha_n(x)-xL^\alpha_n(x)+\alpha L^{\alpha-1}_{n+1}(x) -(n+\alpha)\, L^\alpha_{n-1}(x)$$

$$=(2n+2\alpha+1)\, L^\alpha_n(x)-x\, L^\alpha_n(x)+\alpha\,\{L^\alpha_{n+1}(x)-L^\alpha_n(x)\} -(n+\alpha)\, L^\alpha_{n-1}(x)$$

[Since replacing $n$ by $n+1$ in I we have

$$L^{\alpha-1}_{n+1}(x)=L^\alpha_{n+1}(x)-L^\alpha_n(x)$$

or $$(n+1)\, L^\alpha_{n+1}(x)=(2n+\alpha+1-x)\, L^\alpha_n(x)-(n+\alpha)\, L_{n-1}(x).$$

(III) $x\, L^{\alpha\prime}_n(x)=nL^\alpha_n(x)-(n+\alpha)\, L^\alpha_{n+1}(x).$

**Proof.** From recurrence formula (II) for Laguerre polynomials we have

$$xL_n'(x)=nL_n(x)-nL_{n-1}(x).$$

Replacing $n$ by $(n+\alpha)$, we have

$$xL'_{n+\alpha}(x)=(n+\alpha)\,L_{n+\alpha}(x)+(n+\alpha)\,L_{n-1+\alpha}(x).$$

Differentiating both sides $\alpha$ times w.r.t. '$x$', we have

$$x\,\frac{d^\alpha}{dx^\alpha}\,L'_{n+\alpha}(x)+\alpha.1.\frac{d^{\alpha-1}}{dx^{\alpha-1}}\,L'_{n+\alpha}(x)$$

$$--(n+\alpha)\,\frac{d^\alpha}{dx^\alpha}\,L_{n+\alpha}(x)-(n+\alpha)\,\frac{d^\alpha}{dx^\alpha}\,L_{n-1+\alpha}(x)$$

or $\quad x.(-1)^\alpha\,\dfrac{d^\alpha}{dx^\alpha}\,L'_{n+\alpha}(x)$

$$=n\,(-1)^\alpha\,\frac{d^\alpha}{dx^\alpha}\,L_{n+\alpha}(x)-(n+\alpha)\,(-1)^\alpha\,\frac{d^\alpha}{dx^\alpha}\,L_{n-1-\alpha}(x)$$

or $\quad \mathbf{x\,L_n^{\alpha'}(x)=n\,L_n^\alpha(x)-(n+\alpha)\,L_{n-1}^\alpha(x).}$ **Proved.**

(IV) $\quad \mathbf{L_n^{\alpha'}(x)=-L_{n-1}^{\alpha+1}(x).}$ **[Kanpur 71]**

**Proof.** We have

$$L_n^\alpha(x)=\sum_{r=0}^{n}(-1)^r\,\frac{(n+\alpha)\,!}{(n-r)\,!\,(\alpha+r)\,!\,r\,!}\,x^r$$

$$\therefore \quad L^{\alpha'}_n(x)=\sum_{r=0}^{n}(-1)^r\,\frac{(n+\alpha)\,!\,r}{(n-r)\,!\,(\alpha+r)\,!\,r\,!}\,x^{r-1}$$

$$=\sum_{r=1}^{n}(-1)^r\,\frac{(n+\alpha)\,!}{(n-r)\,!\,(\alpha+r)\,!\,(r-1)\,!}\,x^{r-1}$$

$$=(-1)\sum_{s=0}^{n-1}(-1)^s\,\frac{(n-1+\alpha+1)\,!}{(n-1-s)\,!\,(\alpha+1+s)\,!\,s\,!}\,x^s$$

Taking $r-1=s$

$$=-\,L_{n-1}^{\alpha+1}(x)$$

or $\quad \mathbf{L_n^{\alpha'}(x)=-L_{n-1}^{\alpha+1}(x)}$ **Proved.**

(V) $\quad \mathbf{L_n^{\alpha'}(x)=-\sum_{r=0}^{n-1}L_n^\alpha(x).}$

**Proof.** From recurrence formula III for Laguerre polynomial, we have

$$L_n'(x) = -\sum_{r=0}^{n-1} L_r(x).$$

Replacing $n$ by $(n+\alpha)$ and then differentiatting both sides $\alpha$ times w.r.t. '$x$', we have

$$\frac{d^\alpha}{dx^\alpha} L'_{n+\alpha}(x) = -\sum_{r=0}^{n+\alpha-1} \frac{d^\alpha}{dx^\alpha} L_r(x)$$

$$= -\sum_{r=\alpha}^{n+\alpha-1} \frac{d^\alpha}{dx^\alpha} L_r(x)$$

Since $L_r(x)$ is a polynomial of order $r$, we have $\frac{d^\alpha}{dx^\alpha} L_r(x) = 0$
$0 \leqslant r < \alpha$

$$= -\sum_{r=0}^{n-1} \frac{d^\alpha}{dx^\alpha} L_{s+\alpha}(x)$$ Taking $r = s+\alpha$

or $$(-1)^\alpha \frac{d^\alpha}{dx^\alpha} L'_{n+\alpha}(x) = -\sum_{r=0}^{n-1} (-1)^\alpha \frac{d^\alpha}{dx^\alpha} L_{r+\alpha}(x)$$

or $$L_n^{\alpha'}(x) = -\sum_{r=0}^{n-1} L_n^\alpha(x).$$ **Proved.**

(VI) $$L_n^{\alpha+1}(x) = \sum_{r=0}^{n} L_r^\alpha(x).$$

**Proof.** From recurrence formulae IV and V, we have

$$L_n^{\alpha'}(x) = -L_{n-1}^{\alpha+1}(x)$$

and $$L_n^{\alpha'}(x) = -\sum_{r=0}^{n-1} L_r^\alpha(x).$$

$$\therefore \quad L_{n-1}^{\alpha+1}(x) = \sum_{r=0}^{n-1} L_r^\alpha(x).$$

Replacing $n$ by $n+1$, we have

$$L_n^{\alpha+1}(x) = \sum_{r=0}^{n} L_r^\alpha(x).$$ **Proved.**

## EXAMPLES

**Ex. 1.** *Prove that*

$$\int_x^\infty e^{-t}\, L_n^\alpha(t)\, dt = e^{-x}\left\{L_n^\alpha(x) - L_n^{\alpha'}(x)\right\}.$$

**[Agra 84; Jodhpur 84]**

**Sol.** We have

$$\int_x^\infty e^{-t}.\, L_n^\alpha(t)\, dt = \left[-e^{-t}.\, L_n^\alpha(t)\right]_x^\infty + \int_x^\infty e^{-t}\, L_n^{\alpha\prime}(t)\, dt$$

(Integrating by parts taking $e^{-t}$ as second function)

$$= e^{-x}\, L_n^\alpha(x) + \int_x^\infty e^{-t}.\left[-\sum_{r=0}^{n-1} L_n^\alpha(t).\right].\, dt$$

(from recurrence formula V)

$$= e^{-x}\, L_n^\alpha(x) + \sum_{r=0}^{n-1} \int_x^\infty L_r^\alpha(t)\, dt$$

or $$\int_x^\infty e^{-t} L_n^\alpha(t)\, dt + \sum_{r=0}^{n-1} \int_x^\infty e^{-t}\, L_r^\alpha(t)\, dt = e^{-x}\, L_r^\alpha(x) \qquad \text{...(i)}$$

$$\therefore \quad \sum_{r=0}^{n} \int_x^\infty e^{-t}\, L_r^\alpha(t)\, dt = e^{-x}\, L_n^\alpha(x) \qquad \text{...(ii)}$$

Subtracting (ii) from (i), we have

$$\int_x^\infty e^{-t}\, L_n^\alpha(t)\, dt + \sum_{r=0}^{n-1} \int_x^\infty e^{-t}\, L_r^\alpha(t)\, dt - \sum_{r=0}^{n} \int_x^\infty e^{-t}\, L_r^\alpha(t)\, dt = 0$$

or $$\int_x^\infty e^{-t}\, L_n^\alpha(t)\, dt = \sum_{r=0}^{n} \int_x^\infty e^{-t}\, L_r^\alpha(t)\, dt - \sum_{r=0}^{n-1} \int_x^\infty e^{-t}\, L_r^\alpha(t)\, dt$$

or $$\int_x^\infty e^{-t}\, L_n^\alpha(t)\, dt = e^{-x}\, L_n^\alpha(x) - e^{-x}\, L_{n-1}^\alpha(x) \qquad \text{from (iii)}$$

or $$\int_x^\infty e^{-t}\, \mathrm{L}_n^\alpha(\mathrm{t})\, \mathrm{dt} = e^{-\alpha}\left\{\mathrm{L}_n^\alpha(\mathrm{x}) - e^{-x}\, \mathrm{L}_{n-1}^\alpha(\mathrm{x})\right\}.$$

### Exercise on Chapter VII

1. Show that $\int_x^\infty x^5\, e^{-x}\, L_{10}(x)\, dx = 0.$ [Meerut 74 (S)]
2. Prove that $L_n[0] = n\,!$. [Agra 78, 85, 86]
   [Hint. Proceed as in Ex. 1, taking

$$\sum_{n=0}^{\infty} \frac{1}{n!} t^n L_n(x) = \frac{1}{1-t} e^{-tx/(1-t)}$$

3. Prove that

$L'_n(x) - xL'_{n-1}(x) + n L_{n-1}(x) = 0.$ [Meerut 79 (S), 87]

[Hint. Proceed as in Recc. formula II taking

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) = \frac{1}{1-t} e^{-tx/(1-t)}.$$

4. Prove that $L_n(x) = n!\, {}_1F_1(-n\,;\,1\,;\,x)$. [Meerut 83]

5. Prove that $\displaystyle\int_0^{\infty} e^{-st} . L_n(t)\, dt = \frac{1}{s}\left(1 - \frac{1}{s}\right)^n$. [Raj. 79]

6. Prove that $\displaystyle\int_x^{\infty} e^{-y} L_n(y)\, dy = e^{-x} [L_n(x) - L_{n-1}(x)]$ [Raj. 79]

7. Derive the recurrence relation

$$n L_n^{(\alpha)}(x) = (2n - 1 + \alpha - x) L_{n-1}^{\alpha}(x)$$

$$-(n-1+\alpha) L_{n-2}^{\alpha}(x).$$

[Kanpur 86]

[Hint : Replace $n$ by $n-1$ in Recc, formula III § 7·16].

8. From $L_n^{\alpha}(x) = \dfrac{e^x x^{-\alpha}}{n!} \cdot \dfrac{d^n}{dx^n} (e^{-x} . x^{n+\alpha})$

Prove that $L_1^2(x) = 3 - x$. [Kanpur 85]

9. Find the values of

(i) $\displaystyle\int_0^{\infty} e^{-x} L_3(x) L_5(x)\, dx$ (ii) $\displaystyle\int_0^{\infty} e^{-x} L_4^2(x)\, dx$.

[Rohilkhand 83]

[Ans. (i) 0, (ii) 1]

# 8

# Chebyshev Polynomials

**§ 8·1. Chebyshev's Differential Equation.**

The differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0$$

is called Chebyshev's differential equation.

**§ 8·2. Chebyshev Polynomials.** [Meerut 72, 73, 78]

The Chebyshev polynomials of first kind, $T_n(x)$, and second kind $U_n(x)$ are defind by

$$T_n(x)=\cos(n\cos^{-1}x)$$

and $$U_n(x)=\sin(n\cos^{-1}x)$$

where $n$ is a non-negative integer.

**Note 1.** Sometimes the Chebyshev polynomial of the second kind is defined by

$$u_n(x)=\sin\{(n+1)\cos^{-1}x)\}/\sqrt{[(1-x^2)]}$$

$$=\frac{1}{\sqrt{[(1-x^2)]}}U_{n+1}(x).$$

**2.** Chebyshev's polynomials are also known as Teheibehef, Tbbicheff or Tschebysheff's polynomials.

**§ 8·3. To prove that $T_n(x)$ and $U_n(x)$ are independent solutions of Chebyshev's equation.** [Meerut 78, 79 (S), 80, 82, 89]

**Proof.** Chebyshev's equation is

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0. \qquad \text{...(i)}$$

Now $T_n(x)=\cos(n\cos^{-1}x)$

$$\therefore \quad \frac{d}{dx}T_n(x)=-\sin(n\cos^{-1}x)\, n\frac{-1}{\sqrt{(1-x^2)}}$$

$$=\frac{n}{\sqrt{(1-x^2)}}\sin(n\cos^{-1}x)$$

and $$\frac{d^2}{dx^2}T_n(x)=\frac{nx}{(1-x^2)^{3/2}}\sin(n\cos^{-1}x)-\frac{n^2}{(1-x^2)}\cos(n\cos^{-1}x)$$

$$\therefore \quad (1-x^2)\frac{d^2}{dx^2}T_n(x)-x\frac{d}{dx}T_n(x)+n^2T_n$$

$$=\frac{nx}{\sqrt{(1-x^2)}}\sin(n\cos^{-1}x)-n^2\cos(n\cos^{-1}x)$$

$$-\frac{nx}{\sqrt{(1-x^2)}}\sin(n\cos^{-1}x)+n^2\cos(n\cos^{-1}x)$$

$=0.$ **Proved.**

Hence $T_n(x)$ is the solution of (i).

Similarly we can prove that $U_n(x)$ is the solution of (i), where $U_n(x)=\sin(n\cos^{-1}x)$.

Now we see that $T_n(1)=1$ and $U_n(1)=0$

$\therefore$ $U_n(x)$ can not be written as a constant multiple of $T_n(x)$.

Hence $T_n(x)$ and $U_n(x)$ are independent solutions of (i). **Proved.**

**§ 8·4. To prove that**

(i) $$T_n(x)=\sum_{r=0}^{(n/2)}(-1)^r\frac{n!}{(2r)!\,(n-2r)!}(1-x^2)^r\,x^{n-2r}$$

[Agra 79, 81, 85]

and (ii) $$U_n(x)=\sum_{r=0}^{(n-1)/2}(-1)^r\frac{n!}{(2r+1)!\,(n-2r-1)!}\times(1-x^2)^{r+1/2}.x^{n-2r-1}$$

**Proof.** (i) We have $T_n(x)=\cos(n\cos^{-1}x)$ writing $x=\cos\theta$, we have

$$T_n(x)=\cos n\theta=\tfrac{1}{2}(e^{in\theta}+e^{-in\theta})$$
$$=\tfrac{1}{2}[(\cos\theta+i\sin\theta)^n+(\cos\theta-i\sin\theta)^n]$$
$$=\tfrac{1}{2}[\{x+i\sqrt{(1-x^2)}\}^n+\{x-i\sqrt{(1-x^2)}\}^n]$$

Since $x=\cos\theta$

$\therefore$ $\sqrt{(1-x^2)}=\sin\theta$

$\therefore$ $T_n(x)=\tfrac{1}{2}[\{x+i\sqrt{(1-x^2)}\}^n+\{x-i\sqrt{(1-x^2)}\}^n]$ [Meerut 88]

$$=\tfrac{1}{2}\left\{\sum_{r=0}^{n}{}^nC_r\,x^{n-r}\{i\sqrt{(1-x^2)}\}^r+\sum_{r=0}^{n}{}^nC_r\,x^{n-r}\{-i\sqrt{(1-x^2)}\}^r\right\}$$

(By the binomial theorem)

$$=\tfrac{1}{2}\sum_{r=0}^{n}{}^nC_r\,x^{n-r}\{1+(-1)^r\}.i^r(1-x^2)^{r/2}$$

when $r$ is odd $(-1)^r=-1$ $\therefore$ $1+(-1)^r=0$

and when $r$ is even $1+(-1)^r=2.$

$$\therefore \quad T_n(x)=\frac{1}{2}\sum_{r=0}^{n} {}^nC_r\, x^{n-r}.2\cdot i^r\,(1-x^2)^{r/2}$$

(even $\leqslant n$)

Taking $r=2^s$

when $n$ is integral, $r \leqslant n$ means that $s \leqslant \frac{n}{2}$, $s$ is integral $\therefore \frac{n}{2}$ is greatest integer less than or equal to $n/2$.

We have

$$T(x)=\sum_{s=0}^{(n/2)} {}^n{}_{2s}\, x^{n-2s}.\, i^{2s}.(1-x^2)^s$$

$$=\sum_{s=0}^{(n/2)} \frac{n\,!\,(-1)^s}{(2s)\,!\,(n-2s)\,!}\,(1-x^2)^s\, x^{n-2s}$$

$$=\sum_{r=0}^{(n/2)} (-1)^r \frac{n\,!}{(2r)\,!\,(n-2r)\,!}\,(1-x^2)^r\, x^{n-2r}.$$

**Proved.**

(ii) We have $U_n(x)=\sin(n\cos^{-1}x)$

$=\sin n\theta.$ Putting $x=\cos\theta$

$$=\frac{1}{2i}\,(e^{in\theta}-e^{-in\theta})$$

$$=\frac{1}{2i}\,[(\cos\theta+i\sin\theta)^n-(\cos\theta-i\sin\theta)^n]$$

$$=\frac{1}{2i}\,[\{x+i\sqrt{(1-x^2)}\}^n-\{x-i\sqrt{(1-x^2)}\}^n]$$

$$\mathbf{U}_n(\mathbf{x})=\frac{\mathbf{1}}{\mathbf{2i}}\,[\{\mathbf{x}+\mathbf{i}\sqrt{(\mathbf{1}-\mathbf{x}^2)}\}^n-\{\mathbf{x}-\mathbf{i}\sqrt{(\mathbf{1}-\mathbf{x}^2)}\}^n]$$

$$=\frac{1}{2i}\left\{\sum_{r=0}^{n} {}^nC_r\, x^{n-r}\,\{i\sqrt{(1-x^2)}\}^r-\sum_{r=0}^{n} {}^nC_r\, x^{n-r}\,\{-i\sqrt{(1-x^2)}\}^r\right\}$$

(By binomial theorem)

$$=\frac{1}{2i}\sum_{r=0}^{n} {}^nC_r\, x^{n-r}\,\{1-(-1)^r\}\, i^r\,(1-x^2)^{r/2}$$

when $r$ is even $\quad (-1)^r=1\,;\quad \therefore\quad 1-(-1)^r=0$
and when $r$ is odd $(-1)^r=-1\,;\ 1-(-1)^r=2,$

$$\therefore \quad U_n(x)=\frac{1}{2!}\sum_{r=0}^{n} {}^nC_r.x^{n-r}\,2.i^r\,(1-x^2)^{r/2}$$

(odd $\leqslant n$)

Taking $r=2s+1$ where $s$ is integral $r < n$ means $s < \frac{n-1}{2}$, $s$ is integral ∴ $\frac{n-1}{2}$ is greatest integer less than or equal to $\frac{n-1}{2}$.

We have $U_n(x)=\frac{1}{i}\sum_{s=0}^{(n-1)/2} {}^nC_{2s+1}x^{n-2s-1}\, i^{2s+1}\,(1-x^2)^{s+1/2}$

$$=\sum_{r=0}^{(n-1)/2}(-1)^s\frac{(n)\,!}{(2s+1)\,!\,(n-2s-1)\,!}(1-x^2)^{s+1/2}\,x^{n-2s-1}$$

$$=\sum_{s=0}^{(n-1)/2}(-1)^r\frac{(n)!}{(2r+1)\,!\,(n-2r-1)\,!}(1-x^2)^{r+1/2}\,x^{n-2s-1}$$ **Proved.**

## § 8·5· To find first few Chebyshev Polynomials.

From § 8·4, we have

$$T_n(x)=\sum_{r=0}^{(n/2)}(-1)^r\frac{(n)\,!}{(2r)\,!\,(n-2r)\,!}\;(1-x^2)^r\,x^{n-2r}$$

and $$U_n(x)=\sum_{r=0}^{(n-1)/2}(-1)^r\frac{(n)\,!}{(2r+1)\,!\,(n-2r-1)\,!}(1-x^2)^{r+1/2}\,x^{n-2r-1}$$

Putting $n=0, 1, 2, 3,\ldots$, we have

$T_0(x)=1$

$T_1(x)=x$

$$T_2(x)=\sum_{r=0}^{1}(-1)^r\frac{(2)\,!}{(2r)\,!\,(2-2r)\,!}(1-x^2)^r x.^{2+2r}$$

$$=x^2-(1-x^2)=2x^2-1$$

$$T_3'(x)=\sum_{r=0}^{1}(-1)^r\frac{(3)\,!}{(2r)\,!\,(3-2r)\,!}\cdot(1-x^2)^r\,x^{2-2r}$$

$$=x^3-3.(1-x^2)\,x=4x^3-3x$$

Similarly, $T_4(x)=8x^4-8x^2+1$

$T_5(x)=16x^5-20x^3+5x$ etc.

and $U_0(x)=0$

$U_1(x)=(1-x^2)^{1/2}$

$U_2(x)=\sqrt{(1-x^2)}.2x$

$$U_3(x)=\sum_{r=0}^{1}(-1)^r\frac{(3)!}{(2r+1)!\,(2-2r)!}.(1-x^2)^{r+1/2}\,x^{2-2r}$$

$$=3(1-x^2)^{1/2}.x^2-(1-x^2)^{3/2}.x^0$$

$$=\sqrt{(1-x^2)}.(4x^2-1).$$

Similarly, $U_4(x)=\sqrt{(1-x^2)}\,(8x^3-4x)$

$$U_5(x)=\sqrt{(1-x^2)}\,(16x^4-12x^2+1)$$

**Note.** Obviously $U_n(x)$ is not actually a polynomial, but is a polynomial multiplied by the factor $\sqrt{(1-x^2)}$. While $u_n(x)$ defined in note after § 8·2 is a polynomial of degree $n$.

## § 8·6. Generating Function.

**To prove that**

$$\text{(i)}\ \frac{1-t^2}{1-2tx+t^2}=T_0(x)+2\sum_{n=1}^{\infty}T_n(x).t^n \quad \text{[Meerut 85, 89; Agra 83]}$$

$$\text{and (ii)}\quad \frac{\sqrt{(1-x^2)}}{1-2tx+t^2}=\sum_{n=0}^{\infty}U_{n+1}(x).t^n.$$

**[Meerut 81 (P), 89, Agra 78, 80]**

**Proof.** (i) Putting $x=\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$, we have

$$\frac{1-t^2}{1-2tx+t^2}=\frac{1-t^2}{1-t(e^{i\theta}+e^{-i\theta})+t^2}=\frac{(1-t^2)}{(1-te^{i\theta})\,(1-te^{-i\theta})}$$

$$=(1-t^2)\,(1-t\,e^{-i\theta})^{-1}\,(1-t\,e^{i\theta})^{-1}$$

$$=(1-t^2)\sum_{r=0}^{\infty}(t\,e^{i\theta})^r.\sum_{s=0}^{\infty}(t\,e^{-i\theta})^s$$

(By the binomial theorem)

$$=(1-t^2)\sum_{r,\,s=0}^{\infty}t^{r+s}.e^{i(r-s)\theta}$$

$$=\sum_{r,\,s=0}^{\infty}t^{r+s}.e^{i(r-s)\theta}-\sum_{r,\,s=0}^{\infty}t^{r+s+2}.e^{i(r-s)\theta} \qquad \ldots(1)$$

Now coefficient of $t$ in (i) is

$e^{i(0-0)\theta}=1=T_0(x)$ [Obtained by putting $r=0$, $s=0$ in the first summation]

Again coefficient of $t$ in (1) is

$$e^{i(0-1)\theta}+e^{i(1-0)\theta}$$

[Obtained by putting $r=0$, $s=1$ and $r=1$, $s=0$ in the first summation]

$=e^{-i\theta}+e^{i\theta}=2\cos\theta=2x=2T_1(x).$

In general, the coefficient of $t^n$ in (1) is

$$\sum_{r=0}^{n} e^{i\,[r-(n-r)]\theta}-\sum_{r=0}^{n-2} e^{i\,[r-(n-r-2)]\,\theta}$$

[Obtained by putting $r+s=n$ i.e. $s=n-r$ in the first summation and $r+s+2=n$ i.e. $s=n-r-2$ in the second summation $s\geqslant 0$ gives $n-r\geqslant 0$ i.e. $r\leqslant n$ and $n-r-2\geqslant 0$ i.e. $r\leqslant n-2$]

$$=e^{-in\theta}\sum_{r=0}^{r} e^{i2r\theta}-e^{-i\,(n-2)\theta}\sum_{r=0}^{n-2} e^{i2r\theta}$$

$$=e^{-in\theta}\,[1+e^{2i\theta}+e^{4i\theta}+\ldots+\text{to }(n+1)\text{ terms}]$$
$$-e^{-i(n-2)\theta}\,[1+e^{2i\theta}+e^{4i\theta}+\ldots\text{to }(n-1)\text{ terms}]$$

$$=e^{-in\theta}.\ \frac{1-(e^{2i\theta})^{n+1}}{1-e^{2i\theta}}-e^{-i(n-2)\theta}.\ \frac{1-(e^{2i\theta})^{n-1}}{1-e^{2i\theta}}$$

[The two series within brackets are G.P.]

$$=\frac{e^{-in\theta}-e^{i(n+2)\theta}-e^{-i(n-2)\theta}+e^{in\theta}}{(1-e^{2i\theta})}$$

$$=\frac{e^{-in\theta}\,(1-e^{2i\theta})+e^{in\theta}\,(1-e^{2i\theta})}{(1-e^{2i\theta})}$$

$$=e^{-in\theta}+e^{in\theta}=2\cos n\theta=2\cos(n\cos^{-1}x)=2T_n(x)$$

$\therefore$ from (1), we have $\dfrac{1-t^2}{1-2tx+t^2}=T_0(x)+2\sum_{n=1}^{\infty}T_n(x).t^n.$

(ii) Putting $x=\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$, and simplifying as in case (i), we have

$$\frac{\sqrt{(1-x^2)}}{1-2tx+t^2}=\sin\theta.\ \sum_{r,\,s=0}^{\infty} t^{r+s}.e^{i(r-s)\,\theta} \qquad \ldots(2)$$

$\therefore$ Coefficient of $t^n$ on the R.H.S. is

$$\sin\theta.\ \sum_{r=0}^{n} e^{i\,[r-(n-r)]\,\theta}$$

[Obtained by putting $r+s=n$ i.e. $s=n-r$ $s\geqslant 0$ given $n-r\geqslant 0$ i.e. $r\leqslant n$

$$=\sin\theta.e^{-in\theta}\sum_{r=0}^{n} e^{2ir\theta}$$

$=\sin\theta . e^{-in\theta}.[1+e^{2i\theta}+e^{4i\theta}+ \text{...to } (n+1) \text{ terms}]$

$=\sin\theta . e^{-in\theta}. \dfrac{1-e^{2i\theta\,(n+1)}}{1-e^{2i\theta}}=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}. \dfrac{e^{-in\theta}\,(1-e^{2i\theta(n+1)})}{-e^{i\theta}.(e^{i\theta}-e^{-i\theta})}$

$=\dfrac{e^{-i(n+1)\theta}.(-1+e^{2i\theta(n+1)})}{2i}=\dfrac{e^{i(n+1)\theta}-e^{-i(n+1)\theta}}{2i}$

$=\sin(n+1)\,\theta=\sin\{(n+1)\cos^{-1}x\}=U_{n+1}(x).$

Hence from (2), we have

$$\frac{\sqrt{(1-x^2)}}{1-2tx+t^2}=\sum_{t=0}^{\infty} U_{n+1}(x).t^n. \qquad \textbf{Proved.}$$

## § 8·7. Orthogonal properties of Chebyshev polynomials.

**To prove that**

$$\text{(i)} \quad \int_{-1}^{1} \frac{T_m(x)\,T_n(x)}{\sqrt{(1-x^2)}}\,dx=\begin{cases}0 & m\neq n\\ \pi/2 & m=n\neq 0\\ \pi & m=n=0\end{cases}$$

**[Meerut 74, 76, 77(S), 79, 80(S), 82(P), 83; Agra 80, 82, 87]**

$$\text{and (ii)} \quad \int_{-1}^{1} \frac{U_m(x)\,U_n(x)}{\sqrt{(1-x^2)}}\,dx=\begin{bmatrix}0 & m\neq n\\ \pi/2 & m=n\neq 0\\ 0 & m=n=0\end{bmatrix}$$

**[Meerut 74 (S)]**

**Proof.** We have

$T_m(x)=\cos(m\cos^{-1}x)$

$T_n(x)=\cos(n\cos^{-1}x)$

$$\therefore \int_{-1}^{1} \frac{T_m(x)\,T_n(x)\,dx}{\sqrt{(1-x^2)}}=\int_{-1}^{1} \frac{\cos(m\cos^{-1}x)\cos(n\cos^{-1}x)}{\sqrt{(1-x^2)}}\,dx$$

Putting $x=\cos\theta$ ...(1)

$$\int_0^{\pi} \cos m\theta \cos n\theta \, d\theta$$

$$=\tfrac{1}{2}\int_0^{\pi} [\cos(m+n)\,\theta+\cos(m-n)\,\theta]\,d\theta$$

$$=\tfrac{1}{2}\left[\frac{1}{m+n}\sin(m+n)\,\theta+\frac{1}{m-n}\sin(m-n)\,\theta\right]_0^{\pi}=0$$

if $m\neq n$ i.e. $m-n\neq 0$

If $m=n\neq 0$, from (1), we have

$$\int_{-1}^{+1} \frac{T_m(x)\,T_n(x)}{\sqrt{(1-x^2)}}\,dx=\int_0^{\pi} \cos^2 n\theta\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi} (1+\cos 2n\theta)\,d\theta$$

$$=\frac{1}{2}\left(\theta+\frac{1}{2n}\sin 2n\theta\right)_0^{\pi}=\frac{\pi}{2}.$$

If $m=n=0$ from (1), we have

$$\int_{-1}^{+1} \frac{T_m(x)\,T_n(x)}{\sqrt{(1-x^2)}}\,dx=\int_0^{\pi} d\theta=\pi.$$

**Hence Proved.**

(ii) We have $U_m(x)=\sin(m\cos^{-1}x)$

$U_n(x)=\sin(n\cos^{-1}x)$

$$\therefore\quad \int_{-1}^{1} \frac{U_m(x)\,U_n(x)}{\sqrt{(1-x^2)}}\,dx=\int_{-1}^{1} \frac{\sin(m\cos^{-1}x).\sin(n\cos^{-1}x)}{\sqrt{(1-x^2)}}\,dx$$

Putting $x=\cos\theta$

$$=\int_0^1 \sin m\theta.\sin n\theta\,d\theta \qquad ...(2)$$

Now $\displaystyle\int_0^{\pi} \sin m\theta \sin n\theta\,d\theta$

$$=\frac{1}{2}\int_0^{\pi} [\cos(m-n)\,\theta-\cos(m+n)\,\theta]\,d\theta$$

$$=\frac{1}{2}\left[\frac{1}{m-n}\sin(m-n)\,\theta-\frac{1}{m+n}\sin(m+n)\,\theta\right]_0^{\pi}$$

$=0$ if $m-n\neq0$

*i.e.* $m\neq n$

If $m=n\neq0$, from (2), we have

$$\int_{-1}^{1} \frac{U_m(x)\,U_n(x)}{\sqrt{(1-x^2)}}\,dx=\int_0^{\pi} \sin^2 n\theta\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi} (1-\cos 2n\theta)\,d\theta$$

$$=\frac{1}{2}\left(\theta-\frac{1}{2n}\sin 2n\theta\right)_0^{\pi}=\frac{\pi}{2}.$$

If $m=n=0$, from (2), we have

$$\int_{-1}^{1} \frac{U_m(x)\,U_n(x)}{\sqrt{(1-x^2)}}\,dx=0.$$

**Hence Proved.**

## § 8·8. Recurrence formulae for $T_n(x)$ and $U_n(x)$.

**(I) $T_{n+1}(x)-2x\,T_n(x)+T_{n-1}(x)=0.$** **[Meerut 74 (S), 77]**

**Proof.** We have $T_n(x)=\cos(n\cos^{-1}x)$

$=\cos n\theta$ Putting $x=\cos\theta$

$\therefore\quad T_{n+1}(x)-2xT_n(x)+T_{n-1}(x)$

$=\cos(n+1)\,\theta-2\cos\theta\cos n\theta+\cos(n-1)\,\theta$

$=\cos(n+1)\,\theta-[\cos(n+1)\,\theta+\cos(n-1)\,\theta]+\cos(n-1)\,\theta$

$=0.$

**In short $T_{n+1}-2x\,T_n+T_{n-1}=0$**

**(II) $(1-x^2)\,T'_n(x)=-nx\,T_n(x)+nT_{n-1}(x)$.**

**[Meerut 72, 73, 75, 79, 80, 81, 85, 90; Agra 82]**

**Proof.** We have $T_n(x)=\cos(n\cos^{-1}x)$

$=\cos n\theta$ Putting $x=\cos\theta]$

Also $T_n(x)=\frac{d}{dx}\cos(n\cos^{-1}x)$

$$=-\sin(n\cos^{-1}x).\frac{-n}{\sqrt{(1-x^2)}}$$

$$=\frac{n}{\sqrt{(1-x^2)}}\sin(n\cos^{-1}x)$$

$$=\frac{n}{\sin\theta}\sin n\theta \quad \text{Putting } x=\cos\theta$$

$\therefore$ Putting $x=\cos\theta$

$$(1-x^2)\,T'_n(x)=(1-\cos^2\theta)\frac{n}{\sin\theta}\sin n\theta$$

$$=n\sin\theta\sin n\theta$$

and $-nx\,T_n(x)+n\,T_{n-1}(x)$

$=-n\cos\theta\cos n\theta+n\cos(n-1)\theta.$

$=-n\cos\theta\cos n\theta+n(\cos n\theta\cos\theta+\sin n\theta\sin\theta)$

$=n\sin\theta\sin n\theta$

$\therefore$ $\mathbf{(1-x^2)\,T'_n(x)=-nxT_n(x)+nT_{n-1}(x)}$

**(III)** $\mathbf{U_{n+1}(x)-2xU_n(x)+U_{n-1}(x)=0}$ **[Agra 79]**

**Proof.** We have $U_n(x)=\sin(n\cos^{-1}x)$

$=\sin n\theta$ when $x=\cos\theta$

$\therefore$ Putting $x=\cos\theta$

$U_{n+1}(x)-2xU_n(x)+U_{n-1}(x)$

$=\sin(n+1)\theta-2\cos\theta\sin n\theta+\sin(n-1)\theta$

$=\sin(n+1)\theta-[\sin(n+1)\theta+\sin(n-1)\theta]+\sin(n-1)\theta$

$=0.$

**In short.** $\mathbf{U_{n+1}-2xU_n+U_{n-1}=0.}$

**(IV)** $\mathbf{(1-x^2)\,U'_n(x)=-nxU_n(x)+nU_{n-1}(x).}$

**Proof.** We have $U_n(x)=\sin(n\cos^{-1}x)$

$=\sin n\theta$ Putting $x=\cos\theta$

Also $U'_n(x)=\frac{d}{dx}\sin(n\cos^{-1}x)$

$$=\cos(n\cos^{-1}x).\frac{-n}{\sqrt{(1-x^2)}}$$

$$=-\frac{n}{\sin\theta}\cos n\theta \quad \text{Putting } x=\cos\theta$$

$\therefore$ Putting $x=\cos\theta$

$$(1-x^2)\,U'_n(x)=(1-\cos^2\theta)\frac{-n}{\sin\theta}\cos n\theta$$

$= -n \sin\theta \cos n\theta$

$= n(\cos\theta \sin n\theta - \sin\theta \cos n\theta) - n\cos\theta \sin n\theta.$

$= nU_{n-1}(x) - nx\, U_n(x).$

$\therefore \quad (1-x^2)\, U'_n(x) = -nxU_n(x) + nU_{n-1}(x).$

## EXAMPLES

**Ex. 1.** *Prove that*

(*i*) $T_n(1)=1$ (*ii*) $T_n(-1)=(-1)^n$

(*iii*) $T_{2n}(0)=(-1)^n$ (*iv*) $T_{2n+1}(0)=0.$

**Proof.** We have $T_n(x)=\cos(n\cos^{-1}x)$ ...(1)

(i) Putting $x=1$ in (1), we have

$T_n(1)=\cos(n\cos^{-1}1)=\cos 0=1.$

(ii) Putting $x=-1$ in (1), we have

$T_n(-1)=\cos\{n\cos^{-1}(-1)\}=\cos n\pi=(-1)^n.$

Now putting $x=0$, in (1), we have

$$T_n(0)=\cos(n\cos^{-1}0)=\cos\frac{n\pi}{2}$$

(iii) $\therefore$ if $n$ is even say $n=2m$

$T_{2m}(0)=\cos m\pi=(-1)^m$

$T_{2n}(0)=(-1)^n$

and (iv) if $n$ is odd say $n=2m+1$

$$T_{2m+1}(0)=\cos(2m+1)\frac{\pi}{2}=\cos\left(m\pi+\frac{\pi}{2}\right)$$

$$=(-1)^m.\cos\frac{\pi}{2}=0.$$

$\therefore \quad T_{2n+1}(0)=0.$

**Ex. 2.** *Prove that* (*i*) $U_n(1)=0$

(*ii*) $U_n(-1)=0$

(*iii*) $U_{2n}(0)=0$

*and* (*iv*) $U_{2n+1}(0)=(-1)^n.$

**Proof.** We have $U_n(x)=\sin(n\cos^{-1}x)$ ...(1)

(i) Putting $x=1$ in (1), we have

$U_n(1)=\sin(n\cos^{-1}1)=\sin 0=0.$

(ii) Putting $x=-1$ in (1), we have

$U_n(-1)=\sin\{n\cos^{-1}(-1)\}=\sin n\pi=0,$

Now putting $x=0$ in (1), we have

$$U_n(0)=\sin(n\cos^{-1}0)=\sin\frac{n\pi}{2}.$$

(iii) If $n$ is even, say $n=2m$

$U_{2m}(0)=\sin m\pi=0.$

$\therefore \quad U_{2n}(0)=0.$

and (iv). If $n$ is odd, say $n=2m+1$

$$U_{2m+1}(0)=\sin\,(2m+1)\,\frac{\pi}{2}=\sin\left(m\pi+\frac{\pi}{2}\right)$$

$$=(-1)^m \sin\frac{\pi}{2}=(-1)^m$$

$\therefore \quad U_{2n+1}\,(0)=(-1)^n.$

**Ex. 3.** *Prove that* $T_n(x)=\dfrac{n}{\sqrt{(1-x^2)}}\,U_n(x).$

**Proof.** We have $T_n(x)=\cos\,(n\cos^{-1}x)$

and $\qquad U_n(x)=\sin\,(n\cos^{-1}x)$

$$\therefore \quad T'_n(x)=\frac{d}{dx}\cos\,(n\cos^{-1}x)=-\sin\,(n\cos^{-1}x).\frac{-n}{\sqrt{(1-x^2)}}$$

$$=\frac{n}{\sqrt{(1-x^2)}}\sin\,(n\cos^{-1}x)$$

$$=\frac{n}{\sqrt{(1-x^2)}}\,U_n\,(x).$$

**Proved.**

**Ex. 4.** *Prove that*

$$[T_n(x)]^2-T_{n+1}(x)\,T_{n-1}(x)=1-x^2.$$

**Proof.** We have $T_n(x)=\cos\,(n\cos^{-1}x)$

$=\cos n\theta.$ when $x=\cos\theta$

$\therefore \quad [T_n(x)]^2-T_{n+1}(x)\,T_{n-1}(x)$

$=\cos^2 n\theta-\cos\,(n+1)\,\theta\cos\,(n-1)\,\theta$

$=\cos^2 n\theta-(\cos^2 n\theta-\sin^2\theta)=\sin^2\theta$

$=1-\cos^2\theta=1-x^2.$

**Proved.**

**Ex. 5.** *Show that* $\dfrac{1}{\sqrt{(1-x^2)}}\,U_n(x)$ *satisfies the differential equation* $(1-x^2)\,\dfrac{d^2y}{dx^2}-3x\,\dfrac{dy}{Jx}+(n^2-1)\,y=0.$

**Proof.** Let $y=\dfrac{1}{\sqrt{(1-x^2)}}\,U_n(x)$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{\sqrt{(1-x^2)}}\,U'_n(x)+\frac{x}{(1-x^2)^{3/2}}\,U_n(x)$$

and
$$\frac{d^2y}{dx^2}=\frac{1}{\sqrt{(1-x^2)}}\,U''_n(x)+\frac{2x}{(1-x^2)^{3/2}}\,U'_n(x)$$

$$+\frac{1}{(1-x^2)^{3/2}}\,U_n(x)+\frac{3x^2}{(1-x^2)^{5/2}}\,U_n(x).$$

$$\therefore \quad (1-x^2\,\frac{d^2y}{dx^2}-3x\,\frac{dy}{dx}+(n^2-1)\,y$$

$$=(1-x^2)\left[\frac{1}{\sqrt{(1-x^2)}}U''_n(x)+\frac{2x}{(1-x^2)^{3/2}}U_n'(x)\right.$$
$$\left.+\frac{1}{(1-x^2)^{3/2}}U_n(x)+\frac{3x^2}{(1-x^2)^{5/2}}U_n(x)\right]$$
$$-3x\left[\frac{1}{\sqrt{(1-x^2)}}U'_n(x)+\frac{x}{(1-x^2)^{3/2}}U_n(x)\right.$$
$$\left.+\frac{(n^2-1)}{\sqrt{(1-x^2)}}U_n(x)\right]$$
$$=\frac{1}{\sqrt{(1-x^2)}}[(1-x^2)U''_n(x)-xU'_n(x)+n^2U_n(x)]=0.$$

Since $U_n(x)$ is the solution of

$$(1-x^2\{\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0.$$ **Proved.**

**Ex. 6.** *Show that* $\sqrt{(1-x^2)}\,T_n(x)=U_{n+1}(x)-xU_n(x)$.

**Proof.** We have

$T_n(x)=\cos(n\cos^{-1}x)=\cos n\theta$ where $x=\cos\theta$

and $U_n(x)=\sin(n\cos^{-1}x)=\sin n\theta$

$\therefore$ $\sqrt{(1-x^2)}\,T_n(x)=\sin\theta\cos n\theta$

$=(\sin\theta\cos n\theta+\cos\theta\sin n\theta)-\cos\theta\sin n\theta$

$=\sin(n+1)\theta-\cos\theta\sin n\theta$

$=U_{n+1}(x)-xU_n(x)$. **Proved.**

**Ex. 7.** *Show that* $\sqrt{(1-x^2)}\,U_n(x)=xT_n(x)-T_{n+1}(x)$.

**Proof.** We have

$T_n(x)=\cos(n\cos^{-1}x)=\cos n\theta$ where $x=\cos\theta$

$U_n(x)=\sin(n\cos^{-1}x)=\sin n\theta$

$\therefore$ $\sqrt{(1-x^2)}\,U_n(x)=\sin\theta\sin n\theta$

$=\cos\theta\cos n\theta-(\cos\theta\cos n\theta-\sin\theta\sin n\theta)$

$=\cos\theta\cos n\theta-\cos(n+1)\theta=xT_n(x)-T_{n+1}(x)$. **Proved.**

**Ex. 8.** *Show that*

$$\sum_{r=0}^{n}T_{2r}(x)=\frac{1}{2}\left[1+\frac{U_{2n+1}(x)}{\sqrt{(1-x^2)}}\right]$$

**Proof.** Putting $x=\cos\theta$, we have

$T_n(x)=\cos(n\cos^{-1}x)=\cos n\theta$

and $U_n(x)=\sin(n\cos^{-1}x)=\sin n\theta$

$$\therefore\quad \sum_{r=0}^{n}T_{2r}(x)=\sum_{r=0}^{n}\cos 2r\theta$$

$$=\text{R.P. of }\sum_{r=0}^{n}e^{2i\theta r}$$

$$=\text{R.P. of } [1+e^{2i\theta}+e^{4i\theta}+\ldots+(n+1) \text{ terms}]$$

$$=\text{R.P. of } \frac{1-e^{2(n+1)i\theta}}{1-e^{2i\theta}}$$

$$=\text{R.P. of } \frac{(1-e^{2i(n+1)\theta})(1-e^{-2i\theta})}{(1-e^{2i\theta})(1-e^{-2i\theta})}$$

$$=\text{R.P. of } \frac{\{1-\cos 2(n+1)\theta-i\sin 2(n+1)\theta\}(1-\cos 2\theta+i\sin 2\theta)}{1-(e^{2i\theta}+e^{-2i\theta})+1}$$

$$=\frac{(1-\cos 2\theta)\{1-\cos 2(n+1)\theta\}+\sin 2\theta \sin 2(n+1)\theta}{2-2\cos 2\theta}$$

$$=\frac{1}{2}\left[1-\cos 2(n+1)\theta+\frac{\sin 2\theta \sin 2(n+1)\theta}{1-\cos 2\theta}\right]$$

$$=\frac{1}{2}\left[1-\cos 2(n+1)\theta+\frac{2\sin\theta\cos\theta\sin 2(n+1)\theta}{2\sin^2\theta}\right]$$

$$=\frac{1}{2}\left[1-\cos 2(n+1)\theta+\frac{\cos\theta\sin 2(n+1)\theta}{\sin\theta}\right]$$

$$=\frac{1}{2}\left[1+\frac{\sin 2(n+1)\theta\cos\theta-\sin\theta\cos 2(n+1)\theta}{\sin\theta}\right]$$

$$=\frac{1}{2}\left[1+\frac{\sin(2n+1)\theta}{\sin\theta}\right]$$

$$=\frac{1}{2}\left[1+\frac{U_{2n+1}(x)}{\sqrt{(1-x^2)}}\right]$$

**Proved.**

**Exercise on Chapter VIII**

1. Prove that $2[T_n(x)]^2=1+T_{2n}(x)$.

2. Prove that $\int_{-1}^{+1} x^6 (1-x^2)^{-1/2} T_8(x)\,dx=0$. **[Meerut 74]**

3. Prove that $T_{m+n}(x)+T_{m-n}(x)=2T_m(x)\,T_n(x)$.

4. Prove that $T_n(x)-2x\,T_{n-1}(x)+T_{n-2}(x)=0$. **[Meerut 83 (P)]**

   [Proceed as in Recc. Formula I on page 178]

5. Prove the following

   $$T_n(x)={}_2F_1\left(-n, n; \tfrac{1}{2}; \frac{1-x}{2}\right).$$ **[Meerut 86(R)]**

# 9

# Orthogonal Set of Functions

### § 9·1. Introduction.

The concept of an orthogonal set of functions is a natural generalization of the concept of an orthogonal set of vectors, *i.e.*, a set of mutually perpendicular vectors. In fact, a function can be considered as a generalized vector so that fundamental properties of the set of functions are suggested by the analogous properties of the set of vectors.

**Inner Product.** The **inner product** of two functions $g_1(x)$ and $g_2(x)$ is the number defined by the equation

$$(g_1, g_2)=\int_a^b g_1(x)\, g_2(x)\, dx. \quad [\text{On interval } a \leqslant x \leqslant b].$$

**Orthogonal Function.**

The condition that the two functions be **orthogonal** is written as

$$(g_1, g_2)=0$$

that is

$$\int_a^b g_1(x)\, g_2(x)\, dx=0.$$

### § 9·2. Definitions.

**Orthogonal Set of Functions.** **[Meerut 70, 72, 75, 87]**

A set of real functions $g_1, g_2, g_3 \ldots$ is orthogonal on an interval $a \leqslant x \leqslant b$ if

$$\int_a^b g_m(x)\, g_n(x)\, dx=0 \qquad \text{whenever } m \neq n$$

and is called an **orthogonal set of functions** on that interval.

**Norm.** The **norm** of the function $g(x)$ is defined as the non-negative number

$$\| g \|=\left\{\int_a^b [g(x)]^2\, dx\right\}^{1/2}.$$

Some authors define the norm of $g(x)$ as

$$\| g \|=\int_a^b [g(x)]^2\, dx$$

**Orthonormal set of functions.** [Meerut 76]

Let $\{g_m(x)\}$ $(m=1, 2 \ldots)$ be a orthogonal set of functions. If none of the function $g_m$ have zero norms, each function $g_m$ can be normalized by dividing it by the positive constant $\| g_m \|$

The new set $\{\phi_m(x)\}$ so formed where

$$\phi_m(x) = \| g_m \|^{-1} g_m(x) \qquad [m=1, 2 \ldots]$$

is called an **orthonormal set of functions** on the interval

$$a \leqslant x \leqslant b.$$

Clearly we have

$$\int_a^b \phi_m(x)\, \phi_m(x)\, dx = \begin{cases} 0 & \text{when } m \neq n \\ 1 & \text{when } m = n \end{cases} \quad (m=1, 2, \ldots)$$

Obviously, from an **orthogonal set** we may obtain **an orthonormal** set by dividing each function by its norm on the interval under consideration. The method of orthonormalization is called **Gram-Schmidt orthonormalization method.**

**Ex. 1.** *Show that the functions $g_m = \cos mx$ ; $m=0, 1, 2, \ldots$ form orthogonal set of functions on the interval $-\pi \leqslant x \leqslant \pi$, and determine the corresponding orthonormal set of functions.*

[Meerut 80]

**Sol.** Here $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \, dx \quad m \neq n$

$$= 2\int_0^{\pi} \cos mx \cos nx \, dx$$

$$= \int_0^{\pi} [\cos (m+n)\, x - \cos (m-n)\, x]\, dx$$

$$= \left[ \frac{\sin (m+n)\, x}{m+n} - \frac{\sin (m-n)\, x}{m-n} \right]_0^{\pi} = 0$$

Hence the given functions form an orthogonal set.

Now norm of $g_m$ is

$$\| g_m \| = \| \cos mx \| = \left[ \int_{-\pi}^{\pi} \cos^2 mx \, dx \right]^{1/2}$$

$$= \left[ 2 \int_0^{\pi} \cos^2 mx \, dx \right]^{1/2}$$

$$= \sqrt{(2\pi)}, \qquad \text{when } m=0$$

and $\qquad \| \cos mx \| = \sqrt{\pi} \qquad$ when $m=1, 2, \ldots$

Hence the orthonormal set is

$$\frac{1}{\sqrt{(2\pi)}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\cos 3x}{\sqrt{\pi}}, \ldots$$

Ans.

**Ex 2.** *Show that the set of functions* $\left\{ \sin \dfrac{n\pi x}{c} \right\}$, $(n=1, 2, \ldots)$ *is orthogonal on the interval* $(0, c)$ *and find the orthonormal set.*

[Meerut 87]

**Sol.** Here $\int_0^c \sin\frac{m\pi x}{c}\sin\frac{n\pi x}{c}\,dx \qquad m\neq n$

$$=\frac{1}{2}\left[\int_0^c \cos(m-n)\frac{\pi x}{c}-\cos(m+n)\frac{\pi x}{c}\right]dx$$

$$=\frac{1}{2}\left[\frac{c}{\pi(m-n)}\sin(m-n)\frac{\pi x}{c}-\frac{c}{\pi(m+n)}\sin(m+n)\frac{\pi x}{c}\right]_0^c$$

$=0.$

Hence the given set of functions is orthogonal.

Now

$$\left\|\sin\frac{n\pi x}{c}\right\|=\left\{\int_0^c \sin^2\frac{n\pi x}{c}\,dx\right\}^{1/2}$$

$$=\frac{1}{\sqrt{2}}\left\{\int_0^c\left(1-\cos\frac{2n\pi x}{c}\right)dx\right\}^{1/2}$$

$$=\frac{1}{\sqrt{2}}\left[\left\{x-\frac{c}{2n\pi}\sin\frac{2n\pi x}{c}\right\}_0^c\right]^{1/2}=\sqrt{\left(\frac{c}{2}\right)}$$

Hence ortho-normal set is $\left\{\sqrt{\left(\frac{2}{c}\right)}\sin\frac{n\pi x}{c}\right\}$

$(0\leqslant x\leqslant c,\ n=1, 2, \ldots)$

**Ex. 3.** *Show that the functions* $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$ *form an orthogonal set on an interval* $-\pi\leqslant x\leqslant\pi$ *and obtain the orthogonal set.* [Meerut 72, 76, 81]

**Sol.** Here $\int_{-\pi}^{\pi}\cos mx\sin nx\,dx$ for all $m=0, 1, 2, \ldots$

$n=1, 2, 3, \ldots$ anp $m\neq n$

$$=\frac{1}{2}\int_{-\pi}^{\pi}\{\sin(m+n)x\pm\sin(n-m)x\}\,dx$$

according as $n >$ or $< m$

$$=\frac{1}{2}\left[-\frac{\cos(m+n)x}{(m+n)}\mp\frac{\cos(n-m)x}{n-m}\right]_{-\pi}^{\pi}=0.$$

Also $\int_{-\pi}^{\pi}1.\cos mx\,dx=0,\ \int_{-\pi}^{\pi}\sin^2 mx\,dx=0$

and $\int_{-\pi}^{\pi}\cos^2 mx\,dx=0.$

Hence the functions form orthogonal set.

Now $\|1\|=\left\{\int_{-\pi}^{\pi}1^2\,dx\right\}^{1/2}=\sqrt{(2x)}$

$$\|\cos mx\|=\left\{\int_{-\pi}^{\pi}\cos^2 mx\,dx\right\}^{1/2} \qquad m=1, 2,\ldots$$

$$=\left\{\int_0^{\pi}(1-\cos 2mx)\,dx\right\}^{1/2}=\sqrt{\pi}.$$

and $\|\sin mx\|=\left\{\int_{-\pi}^{\pi}\sin^2 mx\,dx\right\}^{1/2}=\sqrt{\pi}.$

Hence the orthonormal set of functions is

$$\frac{1}{\sqrt{(2\pi)}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}} \ldots$$

**Ans.**

**Ex. 4.** *Show that the functions $f_1(x)=1, f_2(x)=x$ are orthogonal on the interval $(-1, 1)$ and determine the constants A and B so that the function $f_3(x)=1+Ax+Bx^2$ is orthogonal to both $f_1$ and $f_2$ on the interval.* **[Meerut 70; April 74 (S), 78, 79, 82 (P), 84]**

**Sol.**

Here $\displaystyle\int_{-1}^{+1} f_1(x) f_2(x)\, dx=\int_{-1}^{+1} 1.x\, dx=\left(\frac{x^2}{2}\right)_{-1}^{+1}=0.$

Hence $f_1(x)$ and $f_2(x)$ are orthogonal on the interval $(-1, 1)$.

If $f_3(x)=1+Ax+Bx^2$ is orthogonal to both $f_1(x)=1$ and $f_2(x)=x$ on the interval $(-1, 1)$

then $\displaystyle\int_{-1}^{+1} f_1(x) f_3(x)\, dx=0$

or $\displaystyle\int_{-1}^{+1} 1.(1+Ax+Bx^2)\, dx=0$

or $\displaystyle\left(x+\frac{Ax^2}{2}+\frac{Bx^3}{3}\right)_{-1}^{+1}=0$

or $2+(2/3)B=0.$ $\therefore$ $B=-3$

and $\displaystyle\int_{-1}^{+1} f_2(x).f_3(x)=0$

or $\displaystyle\int_{-1}^{+1} (x+Ax^2+Bx^3)\, dx=0$

or $\displaystyle\left(\frac{x^2}{2}+\frac{A}{3}x^3+\frac{B}{4}x^4\right)_{-1}^{+1}=0$

or $(2/3)A=0.$ $\therefore$ $A=0$

Hence the function $f_3(x)$ is orthogonal to both $f_1(x)$ and $f_2(x)$ if $A=0$ and $B=-3$.

**§ 9·3. Generalized Fourier series.** Let

$$\{\phi_n(x)\}, (n=1, 2, 3, \ldots),$$

be an orthogonal set of functions on an interval $(a, b)$. It may be possible to represent an arbitrary function $f$ on the same interval by a linear combination of these functions, in the form

$$f(x)=\sum_{n=1}^{\infty} c_n\phi_n(x)=c_1\phi_1(x)+c_2\phi_2(x)+\ldots+c_n\phi_n(x)+\ldots \qquad \ldots(1)$$

$$(a \leqslant x \leqslant b)$$

If the series (1) converges and represents $f(x)$, it is called a *generalized Fourier Series* of $f(x)$, and its coefficients are called the *Fourier constants* of $f(x)$ with respect to the orthogonal set $\{\phi_n\}$

Multiplying both sides of (1) by $\phi_n(x)$ and then integrating over the interval $a \leqslant x \leqslant b$ on which the functions are orthonormal, we have

$$c_n = \int_a^b f(x)\, \phi_n(x)\, dx.$$

**Note.** If $\{g_n(x)\}$, $(n=1, 2, 3, \ldots)$, is an orthonormal set on an interval $(a, b)$, and

$$f(x) = \sum_{n=0}^{\infty} c_n g_n(x) = c_1 g_1(x) + c_2 g_2(x) + \ldots + c_n g_n(x) + \ldots$$

then
$$c_n = \frac{1}{\| g_m \|^2} \int_a^b f(x)\, g_m(x)\, dx.$$

## § 9·4. Other type of orthogonality.

Some important set of real functions, $g_1, g_2 \ldots$ occurring in applications, are not orthogonal but have the property that for some function $p(x)$

$$\int_a^b p(x)\, g_m(x)\, g_n(x)\, dx = 0. \qquad \text{when } m \neq n$$

Such a set is then said to be *orthogonal* with respect to the *weight* function $p(x)$ on the interval $a \leqslant x \leqslant b$.

The norm of $g_m$ is now defined as

$$\| g_m \|^2 = \int_a^b p(x)\, | g_m(x) |^2\, dx$$

and if the norm of each function $g_m$ is 1, the set is said to be *orthonormal* on that interval with respect to $p(x)$.

The above type of orthogonality is reduced to the ordinary type by using the products $\sqrt{[p(x)]}\, g_n(x)$ as the functions of the set.

## § 9·5. Strum Liouville Equation.

Various important orthogonal sets of functions arise in solutions of second-order differential equations of the form

$$[R(x)\, y']' + [Q(x) + \lambda P(x)]\, y = 0. \qquad \ldots(1)$$

On some interval $a \leqslant x \leqslant b$ satisfying conditions of the form

$$\left.\begin{aligned} &(a) \quad a_1 y + a_2 y' = 0 \quad \text{at} \quad x = a \\ \text{and} \quad &(b) \quad b_1 y + b_2 y' = 0 \quad \text{at} \quad x = b \end{aligned}\right\} \qquad \ldots(2)$$

here $\lambda$ is a real parameter and $a_1, a_2, b_1, b_2$, are given real constants at least one in each conditions (2) being different from zero.

The equation (1) is known as the *Strum-Liouville equation.*

We may see that Legender's equation, Bessel equation and other important equations can be written in the form (1). Conditions (2) are called the boundary conditions.

**Strum-Liouville Problem.** The boundary value problem given by (1), (2) is called a *Strum-Liouville problem*.

[Meerut 73(S), 74(S), 76, 78, 82(P), 83, 86, 87, 88]

The solution $y \neq 0$—are called the *characteristic* functions or *eigen functions* of the problem, and the values of $\lambda$ for which such solution exists, are called characteristic values or *eigen values of the* problem, [Amritsar 80; Meerut 72, 73(S), 77, 87, 88]

**§ 9·6. Theorem.** *Let the function P, Q, R in the Strum-Liouville equation* (1) *be real and continuous on the interval* $a \leqslant x \leqslant b$. *Let* $y_m(x)$ *and* $y_n(x)$ *be given functions of Strum-Liouville problem corresponding to distinct eigen values* $\lambda_m$ *and* $\lambda_n$ *respectively, and let the derivatives* $y'_m(x)$, $y_n'(x)$ *be continuous on the interval. Then* $y_m$ *and* $y_n$ *are orthogonal on that interval with respect to the weight function P.* [Bombay 76; Meerut 71, 73(S), 74(S), 79, 80(S), 82(P), 83, 87; Poona 70; Amritsar 80]

**Proof.** $\because$ $y_m(x)$ and $y_n(x)$ are the solutions of the Strum-Liouville problem

$$[R(x)y']'+[Q(x)+\lambda P(x)]\, y=0 \qquad \text{(i)}$$

on some interval $a \leqslant x \leqslant b$, satisfying conditions

$$\text{(a)} \quad a_1y+a_2y'=0 \qquad \text{at} \quad x=a$$

and

$$\text{(b)} \quad b_1y+b_2y'=0 \qquad \text{at} \quad x=b$$

$\therefore$ we have

$$(Ry'_m)'+(Q+\lambda_mP)\, y_m=0 \qquad \text{...(ii)}$$

and

$$(Ry'_n)'+(Q+\lambda_nP)\, y_n=0 \qquad \text{...(iii)}$$

Multiplying (ii) by $y_n$ and (iii) by $y_m$ and subtracting we have

$$(\lambda_m-\lambda_n)\, Py_my_n=y_m\,(Ry'_n)'-y_n\,(Ry'_m)'$$

or

$$(\lambda_m-\lambda_n)\, Py_my_n=(Ry'_n\, y_m-Ry_m'\, y_n)'$$

Integrating both sides from $a$ to $b$, we have

$$(\lambda_m-\lambda_n)\int_a^b Py_my_n\, dx=\Big(Ry_n'y_m-Ry_m'y_n\Big)_a^b$$

$$=R(b)\,[y_n'(b)\, y_m(b)-y_m'(b)\, y_n\,(b)]$$

$$-R(a)\,[y_n'(a)\, y_m(a)-y_m'(a)\, y_n\,(a)] \quad \text{.. (iv)}$$

Now five different cases arise.

**Case I.** If $R(a)=R(b)=0$ (each).

$\therefore$ From (iv) we have

$$(\lambda_m-\lambda_n)\int_a^b Py_my_n\,dx=0$$

or $$\int_a^b Py_my_n\,dx=0 \qquad \text{Since } \lambda_m\neq\lambda_n$$

from which it follows that $y_m$ and $y_n$ are orthogonal w.r.t. the weight function $P$.

**Case II.** If $R(a)=0$, but $R(b)\neq 0$

$\therefore$ from (iv) we have

$$(\lambda_m-\lambda_n)\int_a^b Py_my_n\,dx=R(b)\,[y_n'(b)\,y_m(b)-y_m'(b)y_n(b)]. \qquad \text{...(v)}$$

Also from $(b)$ we have

$$b_1y_m(b)+b_2y_m'(b)=0 \qquad \text{...(vi)}$$

and $$b_1y_n(b)+b_2y_n'(b)=0 \qquad \text{...(vii)}$$

If $b_2\neq 0$, then multiplying (vi) by $y_n(b)$ and (vii) by $y_m(b)$ and subtracting, we have

$$b_2\,[y'_m(b)\,y_n(b)-y_n'(b)\,y_m(b)]=0$$

and $$y'_m(b)\,y_n(b)-y_n'(b)\,y_m(b)=0$$

$\therefore$ from (v) we have $\int_a^b Py_my_n\,dx=0.$

*i.e.*, $y_m$ and $y_n$ are orthogonal w.r.t. the weight function $P$.

If $b_2=0$ then assuming $b_1\neq 0$.

then multiplying (vi), by $y'_n(b)$ and (vii) by $y_m'(b)$ and subtracting, we have

$$b_1\,[y_m(b)\,y_n'(b)-y'_m(b)\,y_n(b)]=0$$

$$y_m(b)\,y_n'(b)-y'_m(b)\,y_n(b)=0$$

$\therefore$ from (v), we have $\int_a^b Py_my_n\,dx=0$

$\therefore$ $y_m$ and $y_n$ are orthogonal w.r.t. the weight function $P$.

**Case III.** If $R(b)=0$, but $R(a)\neq 0$

$\therefore$ from (iv), we have

$$(\lambda_m-\lambda_n)\int_a^b Py_my_n\,dx=-R(a)\,[y'_n(a).y_m(a)-y_m'(a)y_n(a)]$$

...(viii)

Also from $(a)$, we have

$$a_1y_m(a)+a_2y'_m(a) \qquad \text{...(ix)}$$

and $$a_1y_n(a)+a_2y_n'(a)=0 \qquad \text{...(x)}$$

If $a_2\neq 0$, then multiplying (ix) be $y_n(a)$ and (x) by $y_m\,(a)$ and subtracting, we have

$$a_2[y_m'(a)\, y_n(a) - y_n'(a)\, y_m(a)] = 0$$

or
$$y_m'(a)\, y_n(a) - y_n'(a)\, y_m(a) = 0$$

$\therefore$ from (viii), we have $\int_a^b P y_m y_n dx = 0$

$\therefore$ $y_m$ and $y_n$ are orthogonal w.r.t. the weight function $P$.

If $a_2 = 0$, then we assume $a_1 \neq 0$.

Then multiplying (ix) by $y_n'(a)$ and $(x)$ by $y_m'(a)$ and subtracting, we have

$$a_1[y_n'(a)\, y_m(a) - y_m'(a)\, y_n(a)] = 0$$

or
$$y_n'(a)\, y_m(a) - y_m'(a)\, y_n(a) = 0$$

$\therefore$ from (viii), we have $\int_a^b P y_m y_n \, dx = 0$

$\therefore$ $y_m$ and $y_n$ are orthogonal w.r.t. the weight function $P$.

**Case IV.** If $R(a) \neq 0$, $R(b) \neq 0$.

Thus using $(a)$ and $(b)$ and proceeding as in case II and III, we have from (iv)

$$\int_a^b P y_m y_n \, dx = 0.$$

**Case V.** If $R(a) = R(b)$

$\therefore$ from (iv), we have

$$(\lambda_m - \lambda_n) \int_a^b P y_m y_n \, dx = R(b)\, [y_n'(b)\, y_m(b) - y_m'(b)\, y_n(b)$$
$$- y'_n(a)\, y_m(a) + y_m'(a)\, y_n(a)] \qquad \text{...(xi)}$$

and
$$a_1 y_m(a) + a_2 y_m'(a) = 0 \qquad \text{...(xii)}$$
$$a_1 y_n(a) + a_2 y_n'(a) = 0 \qquad \text{...(xiii)}$$
$$b_1 y_m(b) + b_2 y_m'(b) = 0 \qquad \text{...(xiv)}$$
$$b_1 y_n(b) + b_2 y_n'(b) = 0 \qquad \text{...(xv)}$$

If $a_1 \neq 0$, $b_1 \neq 0$, multiplying (xii) by $y_n(a)$ and (xiii) by $y_m'(a)$ and subtracting, we have

$$y'_m(a)\, y_n(a) - y_m(a)\, y_n'(a) = 0.$$

Also multiplying (xiv) by $y_n(b)$ and (xv) by $y_m(b)$ and subtracting, we have

$$y_m'(b)\, y_n(b) - y_n'(b)\, y_m(b) = 0$$

$\therefore$ from (xi), we have $\int_a^b P y_m y_n \, dx = 0$.

Hence $y_m$ and $y_n$ are orthogonal w r.t. the weight function $P$.

**Proved.**

**§ 9·7. Theorem : Eigen values of the Strum Liouville problem are all real.**

**[Meerut 72, 74, 75, 78, 80(S), 81(P), 83, 86(R), 86, 89, 90]**

**Proof.** Strum-Liouville problem is

$$(rX')'+(q+\lambda p)\,X=0 \qquad ...(i)$$

on the interval $a \leqslant x \leqslant b$
satisfying the conditions

$$a_1X(a)+a_2X'(a)=0 \qquad ...(ii)$$

and

$$b_1X(b)+b_2X'(b)=0 \qquad ...(iii)$$

Let $X$ be an eigen function corresponding to an eigen value

$$\lambda=\alpha+i\beta$$

where $\alpha$, $\beta$ are real numbers. This eigen function on $X(x)$ will satisfy equations (i), (ii) and (iii) and it may be complex valued function.

Now taking the complex conjugate of all the terms in the above equations (i), (ii) and (iii), we have

$$(r\bar{X}')'+(q+\lambda p)\,\bar{X}=0 \qquad ...(iv)$$

$$a_1\bar{X}(a)+a_2\bar{X}'(a)=0 \qquad ...(v)$$

and

$$b_1\bar{X}(b)+b_2\bar{X}'(b)=0 \qquad ...(vi)$$

which shows that $\bar{X}(x)$ is the eigen function corresponding to the eigen values $\bar{\lambda}=\alpha-i\beta$.

Multiplying (i) by $\bar{X}$ and (iv) by $X$ and subtracing, we hbve

$$(rX')'\bar{X}-(rX')'\,X+(\lambda-\bar{\lambda})\,pX\bar{X}=0$$

or

$$(\lambda-\bar{\lambda}).p(x)X(x)\bar{X}(x)=(r\bar{X}')X-(rX')'\bar{X}$$

$$=(r\bar{X}'X-rX'\bar{X})'$$

Integrating both sides from $a$ to $b$, we have

$$(\lambda-\bar{\lambda})\int_a^b p(x)\,X(x)\,\bar{X}(x)\,dx=\left(r\bar{X}'X-rX'\bar{X}\right)_a^b$$

or

$$(\lambda-\bar{\lambda})\int_a^b p(x)\,X(x)\,\bar{X}(x)\,dx$$

$$=r(b)\,[\bar{X}'(b)\,X(b)-X'(b)\,\bar{X}(b)]$$

$$-r(a)\,[\bar{X}(a)\,X'(a)-X'(a)\,\bar{X}\,(a)]$$

or

$$(\lambda-\bar{\lambda})\int_a^b p(x)\,X(x)\,\bar{X}(x)\,dx=0$$

with the help of equation (ii), (iii), (v) and (vi) as in § 9·6.

But $p(x)>0$ when $a<b$ and $X(x)\,\bar{X}(x)=|\,X(x)\,|^2$

$$\therefore \quad \int_a^b p(x)\,X(x)\,\bar{X}(x)\,dx$$

has a positive value in the given interval

$\therefore \quad \lambda-\bar{\lambda}=0 \quad$ or $\quad 2\, i\beta=0$

or $\quad \beta=0 \quad$ *i.e.*, $\lambda$ is real.

Hence eigen-values of the Strum-Liouville problem are all real.

Proved.

**§. 9·8. Theorem. If r(a) > 0 or r(b) >0, the Sturm Liouville problem can not have two linearly independent eigen functions corresponding to the same eigen value. Also each eigen function can be made real valued by multiplying it by an appropriate non-zero constant.**

**Proof.** The Strum Liouville problem is

$$(rX')'+(q+\lambda p)\,X=0 \quad \ldots(i)$$

on the interval $\quad a \leqslant x \leqslant b$

satisfying the condition $\quad a_1X(a)+a_2X'(a)=0 \quad \ldots(ii)$

and $\quad b_1X(b)+b_2X'(b)=0 \quad \ldots(iii)$

Let $X$ and $Y$ be two eigen functions of the Strum-Liouville problem corresponding to the same eigen value $\lambda$.

Now if $r(a) > 0$.

Let us consider the function $W(x)$ such that

$$W(x)=X(x)\,Y'(a)-Y(x)\,X'(a) \quad \ldots(iv)$$

It may be easily seen that $W(x)$ satisfies the equation

$$(rW')'+(q+\lambda p)\,W=0.$$

Also $W'(a)=[W'(x)]_{x=0}=[X'(x)\,Y'(a)-Y'(x)\,X'(a)]_{x=0}=0$

Now if $a_1 \neq 0$, then from (ii), we have

$$\left.\begin{aligned} a_1\,X(a)+a_2\,X'(a)&=0 \\ \text{and} \quad a_1\,Y(a)+a_2\,Y'(a)&=0 \end{aligned}\right\} \quad \ldots(vi)$$

which gives $\quad a_1\,[X(a)\,Y'(a)-Y(a)\,X'(a)]=0$

or $\quad X(a)\,Y'(a)-Y(a)\,X'(a)=0 \quad$ or $\quad W(a)=0.$

But according to the uniqueness theorem for the solutions of linear differential equations, we see that $W(x)=0$ is the only solution of equation (v), such that $W(a)=W'(a)=0. \quad \ldots(viii)$

Thus $\quad X(x)\,Y'(a)-Y(x)\,X'(a)=0.$

If either $X'(a)\neq 0$ or $Y'(a)\neq 0$ then (vi) implies that the functions $X(x)$ and $Y(x)$ are linearly dependent, *i.e.* one function is a constant multiple of the other function.

If $X'(a)=0$ and $Y'(a)=0$ then from (vi), we have $a_1=0$. In this case we can take the function $V(x)$ such that

$$V(x)=Y(a)\,X(x)-X(a)\,Y(x)$$

and proceeding similarly we shell get $V(x)=0$.

Hence $X(x)$ and $Y(x)$ are linearly dependent.

Now let $X=\alpha+i\beta$ be a complex valued eigen function corresponding to a real eigen-value $\lambda$. Substituting $X=\alpha+i\beta$ in (i), (ii) and (iii) and separating the real and imaginary parts we see that $\alpha$ and $\beta$ are also eigen functions of the Sturm Liouville problem corresponding to the same eigen value $\lambda$. Hence according to the first part one must be equal to constant time the other. Say $\beta=k\alpha$ where $k$ is a constant. $\therefore \quad X=\alpha\ (1+ik)$

$\therefore \quad X$ is real, if $k$ is imaginary constant factor.

If $k$ is real constant then $X$ may be made real by multiplyin it by an appropriate non-zero constant.

Hence each eigen function can be made real valued by multiplying it by an appropriate non-zero constant. **Proved.**

## EXAMPLES

**Ex. 5.** *For the eigen value problem*

$$X''+\lambda X=0,\ X(0)=0,\ X(\pi)=0$$

*obtain the set of eigen functions and the eigen values.*

**[Meerut 70, 80, 81 ; Bombay 76]**

**Solution. Case I.** If $\lambda=0$, we have

$$X''=0 \quad \therefore \quad X(x)=Ax+B \quad \therefore \quad X(0)=0=B$$

and $\quad X(\pi)=0=A\pi+B \quad$ giving $\quad A=0$ and $B=0$

$\therefore \quad X(x)=0$ which is not an eigen function.

**Case II.** If $\lambda=-n^2$, we have $\quad X''-n^2X=0$

$$\therefore \quad X(x)=Ae^{nx}+Be^{-nx} \quad \therefore \quad X(0)=0=A+B$$

and $\quad X(\pi)=0=Ae^{n\pi}+Be^{-n\pi} \quad$ giving $\quad A=0$ and $B=0$

$\therefore \quad X(x)=0$ which is also not an eigen function.

**Case III.** If $\lambda=n^2$, we have $\quad X''+n^2X=0$

$$\therefore \quad X(x)=A\cos nx+B\sin nx, \quad \therefore \quad X(0)=0=A$$

and $\quad X(\pi)=0=B\sin n\pi$

giving $A=0$ and either $B=0$ or $\sin n\pi=0$.

If $B=0$, $X(x)=0$ and which is not eigen value

$\therefore$ taking $\sin n\pi=0$

which gives $\qquad n=\pm 1, \pm 2, \ldots$

$\therefore$ Eigen functions are $X(x)=\sin nx \qquad$ taking $B=1$

and eigen values are $\lambda=n^2$, $n=1, 2, 3, \ldots$ **Ans.**

**Ex. 6.** *Find all the eigen values and eigen functions of the Strum-Lioville problem*

$$X''+\lambda X=0,\ X'(0)=3,\ X'(c)=0.$$

**[Meerut 72, 75, 77, 78, 82 (P), 85]**

**Solution.** **Case I.** If $\lambda=0$, we have $X''=0$

$$\therefore \quad X(x)=Bx+A,\ X'(x)=B, \quad \therefore \quad X'(0)=0=B$$

and $\quad X'(c)=0=B, \quad \therefore \quad X(x)=A$

which is the eigen function and the corresponding eigen value is $\lambda=0$.

**Case II.** If $\lambda=-n^2$, we have

$$X''-n^2X=0 \quad \therefore \quad X(x)=Ae^{nx}+Be^{-nx}$$

$$\therefore \quad X'(0)=0=An-Bn, \qquad X'(x)=Ane^{nx}-Bne^{-nx}$$

and $\quad X'(c)=0=Ane^{nc}-Bne^{-nc}, \qquad$ giving $A=0=B$

$X(x)=0 \qquad$ which is not an eigen function.

**Case III.** If $\lambda=n^2$, we have

$$X''+n^2X=0 \quad \therefore \quad X(x)=A\cos nx+B\sin nx$$

$$\therefore \quad X'(0)=0=Bn \quad \therefore \quad B=0$$

and $\quad X'(c)=-An\sin nc+Bn\cos nc=0$

or $\quad A\sin nc=0$

if $\quad A=0$ then $X(x)=0$, which is not an eigen function.

$\therefore$ taking $\sin nc=0$

$$\therefore \quad nc=m\pi \quad \text{or} \quad n=\frac{m\pi}{c},\ m=1, 2, \ldots$$

$$\therefore \quad X(x)=A\cos\frac{m\pi x}{c}$$

$\therefore$ eigen functions $X(x)=\cos\dfrac{m\pi x}{c}$. Taking $A=1$

and eigen values are $\lambda=\dfrac{m^2\pi^2}{c^2}$, $m=1, 2, 3 \ldots.$ **Ans.**

**Ex. 7.** *Find all eigen values and eigen functions of the Strum-Liouvile problem* $X''+\lambda X=0,\ X(0)=0.\ X'\left(\dfrac{\pi}{2}\right)=0.$ **[Meerut 84]**

**Solution.** **Case I.** If $\lambda=0$, we have, $X''=0$

$$\therefore \quad X(x)=Ax+B, \quad \therefore \quad X(0)=0=B$$

and $\quad X'\left(\dfrac{\pi}{2}\right)=0=A$

$\therefore \quad X(x)=0$, which is not an eigen function.

**Case II.** If $\lambda=-n^2$, we have $X''-n^2X=0$

$\therefore \quad X(x)=Ae^{nx}+Be^{-nx}, \quad \therefore \quad X(0)=0=A+B$

$$X'\left(\frac{\pi}{2}\right)=0=Ane^{n\pi/2}-Bne^{-n\pi/2}$$

giving $\quad A=0=B \quad \therefore \quad X(x)=0$ which is not an eigen function

**Case III.** If $\lambda=n^2$, we have $X''+n^2X=0$

$\therefore \quad X(x)=A\cos nx+B\sin nx \qquad \therefore \quad X(0)=0=A$

and $\quad X'\left(\frac{\pi}{2}\right)=0=Bn\cos\frac{n\pi}{2} \qquad \therefore \quad A=0$

if $B=0$, then $X(x)=0$ which is not an eigen function.

$$\cos\frac{n\pi}{2}=0 \text{ giving } n=(2m-1),\ m=1, 2, 3, \ldots$$

$$\therefore \quad X(x)=B\sin[(2m-1)x]$$

$\therefore$ eigen functions are $X(x)=\sin[(2m-1)x]$

and eigen values are $\lambda=(2m-1)^2$, 1, 2, 3, .. **Ans.**

**Ex. 8.** *Find all the eigen values and eigen functions of the Strum-Liouville problem*

$y''+\lambda y=0,\ y(0)+y'(0)=0,\ y(1)+y'(1)=0.$ **[Meerut 89]**

**Sol. Case I.** If $\lambda=0$, the equation reduces to $y''=0$,

whose solution is $\quad y(x)=Ax+B$

so that $\quad y'(x)=A$

$\therefore \quad y(0)+y'(0)=B+A=0$

and $\quad y(1)+y'(1)=(A+B)+A=0$

solving, we have $A=0$, $B=0$

$y(x)=0$, which is not an eigen function.

**Case II.** If $\lambda=-K^2$, the equation reduces to $y''-K^2y=0$

whose solution is $\quad y(x)=Ae^{Kx}+Be^{-Kx}$

so that $\quad y'(x)=AKe^{Kx}-BKe^{-Kx}$

$\therefore \quad y(0)+y'(0)=A+B+AK-BK=0$

or $\quad A(1+K)+B(1-K)=0$

and $\quad y(1)+y'(1)=Ae^K+Be^{-K}+AKe^K-BKe^{-K}=0$

or $\quad A(1+K)e^K+B(1-K)e^{-K}=0$

Solving we have $\quad A=0,\ K=1 \quad \therefore \quad y(x)=Be^{-x}$

or $\quad B=0,\ K=-1 \quad \therefore \quad y(x)=Ae^{-x}$

$\therefore$ eigen function is $y(x)=e^{-x}$, and eigen value is

$$\lambda=-K^2=-1.$$

**Case III.** If $\lambda=K^2$, the equation reduces to $y''+K^2y=0$

whose solution is $\quad y(x)=A\cos Kx+B\sin Kx$

so that $\quad y'(x)=-AK\sin Kx+BK\cos Kx$

$\therefore \quad y(0)+y'(0)=A+BK=0$

and $\quad y(1)+y'(1)=A\cos K+B\sin K-AK\sin K+BK\cos K=0$

Putting $A=-BK$, we have

$$B(1+K^2)\sin K=0$$

giving $\quad B=0$ or $\sin K=0$

Now if $B=0,\ A=0$

$\therefore\ y(x)=0$, which is not an eigen *function*

$\therefore$ $\sin K=0$, which gives

$$K=n\pi,\ n=1,\ 2,\ 3,\ldots;\ y(x)=A\cos n\pi x+B\sin n\pi x$$

$$A=-BK=-n\pi B \text{ or } y(x)=-Bn\pi\cos n\pi x+B\sin n\pi x$$

$\therefore$ eigen function are

$$y_n(x)=B_n\ (\sin n\pi x-n\pi\cos n\pi x)$$

and eigen values are $\lambda=K^2=n^2\pi^2.\ n=1,\ 2,\ 3,\ldots$ **Ans.**

**Ex. 9.** *Find the eigen values and eigen functions for the problem* $\{xy'(x)\}'+\dfrac{\lambda}{x}y(x)=0,\ y'(1)=y'(e^{2\pi})=0.$

**[Amritsar 80 ; Meerut 90]**

**Sol.** The given equation can be written as

$$xy''(x)+y'(x)+\frac{\lambda}{x}y(x)=0$$

or
$$x^2y''(x)+xy'(x)+\lambda y(x)=0.$$

Changing the independent variable from $x$ to $z$, by the substitution $x=e^z$ so that $x\dfrac{dy}{dx}=\dfrac{dy}{dz}=Dy$

and
$$\frac{x^2\,d^2y}{dx^2}=D\,(D-1)\,y \text{ where } D\equiv\frac{d}{dz}$$

The equation reduces to $\{D\,(D-1)\,y+Dy+\lambda y\}=0$

or
$$D^2y+\lambda y=0 \quad\text{or}\quad \frac{d^2y}{dx^2}+\lambda y=0.$$

**Case I.** When $\lambda=0$, we have $\dfrac{d^2y}{dx^2}=0$

$\therefore\ y=Az+B$ or $y(x)=A\log x+B$

so that
$$y'(x)=\frac{A}{x}$$

$$\therefore\ y'(1)=A=0$$

$\therefore$ Corresponding to the eigen value $\lambda=0$.

$y(x)=B$ is the eigen function.

**Case II.** When $\lambda=-K^2$.

The equation is $\dfrac{d^2y}{dz^2}-K^2y=0$

$$\therefore\ y=Ae^{Kz}+Ce^{-Kz}$$

or
$$y(x)=Ax^K+Bx^{-K}$$

so that
$$y'(x)=AKx^{K-1}-BKx^{-K-1}$$

$$\therefore\ y'(1)=AK-BK=0 \text{ or } B=A$$

and $\quad y'(e^{2\pi}) = AK(e^{2\pi})^{K-1} - BK(e^{2\pi})^{-K-1} = 0$

or $\quad Ae^{2\pi K} - Ae^{-2\pi K} = 0 \quad \therefore \; B = A$

or $\quad A(e^{2\pi K} - e^{-2\pi K}) = 0 \quad \therefore \; A = 0, B = 0$

so that $\quad y(x) = 0$ which is not an eigen function.

**Case III.** When $\lambda = K^2$

The equation is $\dfrac{d^2y}{dz^2} + K^2 y = 0$

$$\therefore \; y = A \cos Kz + B \sin Kz$$

or
$$y(x) = A \cos(K \log x) + B \sin(K \log x)$$

so that
$$y'(x) = -\frac{AK}{x} \sin(K \log x) + \frac{BK}{x} \cos(K \log x)$$

$$\therefore \; y'(1) = BK = 0 \quad \therefore \; B = 0$$

and
$$y'(e^{2\pi}) = -AK\, e^{-2\pi} \sin(K.2\pi) = 0 \quad \because \; B = 0$$

$$\therefore \; \text{either } A = 0 \text{ or } \sin 2\pi K = 0.$$

But $A \neq 0$, otherwise $y(x) = 0$, which is not an eigen function

$$\therefore \; \sin 2\pi K = 0 \qquad \therefore \; 2\pi K = n\pi$$

or
$$K = \frac{n}{2},\; n = 1, 2 \ldots$$

$$\therefore \; \lambda = K^2 = \frac{n^2}{4},\; n = 1, 2, 3 \ldots$$

$$\therefore \; y(x) = A \cos\left(\frac{n}{2} \log x\right)$$

$\therefore$ eigen functions are

$$y_n(x) = A_n \cos\left(\frac{n}{2} \log x\right)$$

and eigen values are $\lambda = \dfrac{n^2}{4}$, $n = 1, 2 \ldots\ldots$ **Ans.**

### § 9·9. Orthogonality of Legendre Polynomials. [Bombay 76]

The Legendre's differential equation.

$$(1 - x^2)\, y'' - 2xy' + n(n+1)\, y = 0$$

may be written as
$$[(1 - x^2)\, y']' + \lambda y = 0 \qquad \ldots(1)$$

where $\lambda = n(n+1)$.

and is therefore a Strum-Liouville equation with

$$R(x) = 1 - x^2,\; P(x) = 1 \text{ and } Q(x) = 0.$$

Here no boundary conditions are needed to form a Strum-Liouville problem on the interval $(-1, 1)$ since $R = 0$ when $x = \pm 1$,

Further we know that Legendre polynomials

$P_n(x)$, $(n=0, 1, 2......)$,

are the solutions of the problem hence they are the eigen functions and since they have continuous derivatives, therefore it follows that $\{P_n(x)\}$, $n=0, 1, 2, ...$, are orthogonal on the interval $-1 \leqslant n \leqslant 1$ with respect to the weight function $P=1$.

i.e.,
$$\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx=0 \qquad (m \neq n)$$

and
$$\| P_m \|^2=\int_{-1}^{+1} P_m^2(x)\, dx=\frac{1}{2m+1} \qquad m=0, 1, 2,......$$

If $g_0(x)$, $g_1(x)$,...are eigen functions which are orthogonal on an interval $a \leqslant x \leqslant b$ with respect to the weight function $p(x)$, and if a given function $f(x)$ can be represented by a generalized Fourier series

$$f(x)=\sum_{n=1}^{\infty} C_n\, g_n(x),$$

then,
$$C_m=\frac{1}{\| g_m \|^2}\int_a^b p(x) f(x)\, g_m(x)\, dx \qquad (m=0, 1, 2...)$$

where
$$\| g_m \|^2=\int_a^b p(x)\, g_m^2(x)\, dx.$$

**§ 9·10. Orthogonality of Bessel Function.** We know that $J_n(v)$ is the solution of the Bessel equation

$$v^2 \frac{d^2y}{dv^2}+v\frac{dy}{dv}+(v^2-n^2)\, y=0 \text{ where } n \text{ is +ve integer}$$

$$\therefore\ v^2 \frac{d^2J_n(v)}{dv^2}+v\frac{dJ_n(v)}{dv}+(v^2-n^2)\, J_n(v)=0.$$

Putting $v=\lambda x$, so that

$$\frac{dJ_n}{dv}=\frac{dJ_n}{dx}\frac{dx}{dv}=\frac{1}{\lambda} J_n', \qquad \text{where } \lambda \text{ is a constant}$$

and
$$\frac{d^2J_n}{dv^2}=\frac{1}{\lambda} J_n''.\frac{dx}{dv}=\frac{1}{\lambda} J_n''.$$

We have $x^2J_n''(\lambda x)+xJ_n'(\lambda x)+(\lambda^2x^2-n^2)\, J_n(\lambda x)=0$ which may be written as

$$[xJ_n'(\lambda x)]'+\left(-\frac{n^2}{x}+\lambda^2 x\right) J_n(x)=0. \qquad ...(1)$$

which is Strum-Liouvile equation for each fixed $n$ with

$$R(x)=x,\ Q(x)=-\frac{n^2}{x} \text{ and } P(x)=x,$$

here parameter being $\lambda^2$ instead of $\lambda$.

Since $R(x)=x$ for $x=0$, it follows from theorem § 9·6, that solutions of (1) on an interval $0 \leqslant x \leqslant a$ satisfying the boundary conditions $J_n(\lambda a)=a$,

form an orthogonal set with respect to the weight function $P(x)=x$.

Let $\alpha_n < \alpha_{2n} < \alpha_{3n}$ donote the positive zeros of $J_n(v)$ therefore (2) holds for

$$\lambda a=a_{mn} \text{ or } \lambda=\lambda_{mn}=\frac{\alpha_{mn}}{a}, \qquad (m=1, 2, \ldots n \text{ fixed})$$

and since $J_n'$ is continuous, also at $x=0$, therefore for each fixed $n=0, 1, 2, \ldots$ the Bessel functions $J_n(\lambda_{mn} x)$ $(m=1, 2, \ldots)$ with $\lambda_{mn}=\dfrac{\alpha_{mn}}{a}$, form an orthogonal set on an interval $0 \leqslant x \leqslant a$ with respect to weight function $p(x)=x$.

$$\therefore \int_0^a xJ_n(\lambda_{mn} x) J_n(\lambda_{pn} x)=0 \qquad (p \neq m)$$

Thus we obtained infinitely many orthogonal sets corresponding to each fixed value of $n$.

If a function is represented by generalised Fourier Bessel series

$$f(x)=\sum_{m=1}^{\infty} C_m J_r(\lambda_{mn} x), \; n \text{ fixed} \qquad \ldots(3)$$

then $C_m=\dfrac{1}{\| J_n(\lambda_{mn} x) \|^2}\displaystyle\int_0^a x f(x) J_n(\lambda_{mn} x)\, dx$, $m=1, 2, \ldots$

Since $p(x)=x$. $\qquad \lambda_{mn}=\dfrac{\alpha_{mn}}{a}$

$$\text{where } \quad \| J_n(\lambda_{mn} x) \|^2=\int_0^a x J_n^2(\lambda_{mn} x)\, dx \qquad \ldots(4)$$

To find $\| (J_n \lambda_{mn}) \|^2$ let us proceed as follows.

Multiplying (1) by $2xJ_n'(\lambda x)$, we have

$$2xJ_n'(\lambda x)\,[xJ_n'(\lambda x)]'+\left(\lambda_{mn}\frac{n^2}{x}+\lambda^2 x\right) 2xJ_n(\lambda x)\, J_n'(\lambda x)=0$$

or
$$\{[xJ_n'(\lambda x)]^2\}'+(\lambda^2 x^2-n^2)\,[J_0^2(\lambda x)]'=0.$$

Integrating over the limits 0 to $a$, we have

$$\Big[\{xJ'_n(\lambda x)\}^2\Big]_0^a=-\int_0^a (\lambda^2 x^2-n^2)\,[J_n^2(\lambda x)]'\, dx$$

Integrating R.H.S. by parts, we have

$$\Big[\{xJ_n'(\lambda x)\}\Big]_0^a=-\Big[(\lambda^2 x^2-n^2)\, J_n^2(\lambda x)\Big]_0^a$$
$$+2\lambda^2\int_0^a xJ_n^2(\lambda x)\, dx. \qquad \ldots(5)$$

From Recurrence formula for $J_n(v)$, we have

$$\frac{d}{dv}\{v^{-n} J_n(v)\} = -v^{-n} J_{n+1}(v)$$

or
$$v^{-n}\frac{d}{dv} J_n(v) - v^{-n-1} J_n(v) = -v^{-n} J_{n+1}(v),$$

$\therefore$ Multiplying both sides by $v^{n+1}$,

$$v\frac{d}{dv} J_n(v) - nJ_n(v) = -v\, J_{n+1}(v)$$

Putting $v = \lambda x$,

$$\lambda x \frac{dJ_n(\lambda x)}{d(\lambda x)} - nJ_n(\lambda x) = -\lambda x J_{n+1}(\lambda x)$$

or
$$\lambda x \frac{dJ_n(\lambda x)}{d(\lambda x)} = n\, J_n(\lambda x) - \lambda x J_{n+1}(\lambda x)$$

or
$$\lambda x \frac{dJ_n(\lambda x)}{dx} \cdot \frac{1}{\lambda} = nJ_n(\lambda x) - \lambda x J_{n+1}(\lambda x)$$

or
$$xJ'_n(\lambda x) = nJ_n(\lambda x) - \lambda x J_{n+1}(\lambda x)$$

Substituting in (5), we have

$$\Big[\{nJ_n(\lambda x) - \lambda x\, J_{n+1}(\lambda x)\}^2\Big]_0^a = -\Big[(\lambda^2 x^2 - n^2)\, J_n^2(\lambda x)\Big]_0^a$$
$$+2\lambda^2 \int_0^a xJ_n^2(\lambda x)\, dx$$

If $\lambda = \lambda_{mn}$, then $J_n(\lambda a) = J_n(\lambda_{mn} a) = 0$

and since $\quad J_n(0) = 0, \quad (n = 1, 2, \ldots),$

$\therefore$ we have

$$\lambda^2_{mn} a^2\, J^2_{n+1}(\lambda_{mn} a) = 2\lambda^2_{mn} \int_0^a xJ_n^2(\lambda_{mn} x)\, dx$$
$$= 2\lambda^2_{mn} \| J_n(\lambda_{mn} x) \|^2$$

[since weight function $p(x) = x$]

$$\| J_n(\lambda_{mn} x) \|^2 = \frac{a^2}{2} J^2_{n+1}(\lambda_{mn} a) = \frac{a^2}{2} J^2_{n+1}(\alpha_{mn})$$

where $\quad \alpha_{mn} = \lambda_{mn} \cdot a$

$$\therefore \quad C_m = \frac{2}{a^2 J^2_{n+1}(\alpha_{mn})} \int_0^a xf(x)\, J_n(\lambda_{mn}\, \alpha)\, dx \qquad \ldots(6)$$

*n* fixed.

$$\lambda_{mn} = \frac{\alpha_{mn}}{a}, \qquad m = 1, 2, \ldots$$

Thus generalised Fourier Bessel Series is given by (3) with the coefficients given by (6).

**§ 9·11. Orthogonality of Hermite Polynomials.**

The Hermite polynomials $H_n(x)$, given by

$$H_n(x)=(-1)^n\, e^{x^2}\, \frac{d^n e^{-x^2}}{dx^n}$$

are orthogonal w.r.t. the weight function $p(x)=e^{-x^2}$ on the interval $-\infty \leqslant x \leqslant \infty$.

$$\int_{-\infty}^{\infty} H_m(x)\, H_n(x)\, e^{-x^2}\, dx=(-1)^n \int_{-\infty}^{\infty} H_m(x).\frac{d^n e^{-x^2}}{dx^n},$$

$$=(-1)^n.\left[H_m(x)\frac{d^{n-1}}{dx^{n-1}}\, e^{-x^2}\right]_{-\infty}^{\infty}$$

$$-(-1)^n \int_{-\infty}^{\infty} H'_m(x)\, \frac{d^{n-1}}{dx^{n-1}}\, e^{-x^2}\, dx$$

$$=-(-1)^n \int_{-\infty}^{\infty} 2m\, H_{m-1}(x)\, \frac{d^{n-1}}{dx^{n-1}}\, e^{-x^2}\, dx$$

[Since $e^{-x^2}$ and all its derivatives vanish for infinite $x$ and $H'_n=2n\, H_{n-1}$].

$$=(-1)^{n-1}.2m \int_{-\infty}^{\infty} H_{m-1}(x)\, \frac{d^{n-1}}{dx^{n-1}}\, e^{-x^2}\, dx$$

[proceeding similarly again and again]

$$=(-1)^{n-m}\, 2^m\, m\,! \int_{-\infty}^{\infty} H_0(x)\, \frac{d^{n-m}}{dx^{n-m}}\, e^{-x^2}\, dx$$

$n > m$

$$=(-1)^{n-m}\, m\,! \int_{-\infty}^{\infty} \frac{d^{n-m}}{dx^{n-m}}\, e^{-x^2}\, dx$$

[Since $H_0(x)=1$]

$$=(-1)^{n-m}.2^m\, m\,!\left(\frac{d^{n-m-1}}{dx^{n-m-1}}\, e^{-x^2}\right)_{-\infty}^{\infty}$$

$$=0.$$

**Note.** See also § 6·7.

**To get the orthogonal system.**

$$\text{Now} \int_{-\infty}^{\infty} H_n^2(x)\, e^{-x^2}\, dx$$

$$=\int_{-\infty}^{\infty} H_n(x).\frac{d^n}{dx^n}\, e^{-x^2}\, dx$$

integrating as above $n$ times.

$$=2^n\, n\,! \int_{-\infty}^{\infty} H_0(x)\, e^{-x^2}\, dx$$

$$=2^n\, n\,!\int_{-\infty}^{\infty} e^{-x^2}\, dx$$

$$=2^n\, n\,!\, 2\int_{0}^{\infty} e^{-x^2}\, dx$$

$$=2^n\, n\,!\, \sqrt{\pi}.$$

The functions of the orthogonal system, are

$$\psi_n(x)=\frac{H_n(x)\, e^{-x^2/2}}{\sqrt{[2^n\, n\,!\sqrt{\pi}]}}. \qquad (\nu=0, 1, 2, \ldots)$$

**§ 9·12. Orthogonality of Laguerre polynomials.**

**[Meerut 88; Kanpur 84]**

The Laguerre polynomials $L_n(x)$ given by

$$L_n(x)=e^x \frac{d^n}{dx^n}(x^n e^{-x})$$

are orthogonal w.r.t. the weight function $p(x)=e^{-x}$ on the interval $0 \leqslant x \leqslant \infty$.

$$\int_0^{\infty} L_m(x).L_n(x)\, e^{-x}\, dx=\int_0^{\infty} L_m(x).\frac{d^n}{dx^n}(x^n e^{-x})\, dx$$

$$=\left[L_m(x)\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\right]_0^{\infty}-\int_0^{\infty} L'_m(x)\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\, dx$$

$$=-\int_0^{\infty} L_m'(x)\frac{d^{n-1}}{dx^{n-1}}(x^n e^{-x})\, dx$$

proceeding similarly

$$=(-1)^m\int_0^{\infty} L_m^m(x).\frac{d^{n-m}}{dx^{n-m}}(x^n e^{-x}\, dx)$$

$$n<m$$

$$=(-1)^m\int_0^{\infty}(-1)^m\, m\,!\,\frac{d^{n-m}}{dx^{n-m}}(x^n e^{-x})\, dx$$

$$=m\,!\left[\frac{d^{n-m-1}}{dx^{n-m-1}}(x^n e^{-x})\right]_0^{\infty}=0.$$

**Note.** See also § 7·7.

**To get the orthogonal system.**

Now $\int_0^{\infty} L^2{}_n(x).e^{-x}\, dx$

$$=\int_0^{\infty} L_n(x)\frac{d^n}{dx^n}(x^n e^{-x})\, dx$$

$$=(-1)^n\int_0^{\infty} L_n^n(x)\,(x^n e^{-x})\, dx$$

$$=(-1)^n(-1)^n\, n\,!\int_0^{\infty} x^n e^{-x}\, dx=(n\,!)^2.$$

Thus the functions of the orthonormal system are

$$\psi_\nu(x)=\frac{e^{-x/2}\,L_\nu(x)}{n\,!}. \qquad (\nu=0,1,2,\ldots)$$

**§ 9·13. Orthogonality of Chebyshev Polynomials.**

Chebyshev Polynomials $T_n(x)$ and $U_n(x)$ are both orthogonal w.r.t. the weight function $1/\sqrt{(1-x^2)}$ on the interval $-1<m<1$.

(See § 8·7)

**§ 9·14. Orthogonality of Jacobi Polynomials.**

The Jacobi polynomials $G_n(x)$ are orthogonal w.r.t. weight function $p(x)=x^{q-1}(1-x)^{p-q}$. $(q>0,\ p-q>-1)$
on the integral $0\leqslant x\leqslant 1$

$$G_n(x)=\frac{x^{1-q}(1-x)^{q-p}}{q(q+1)\ldots(q+n-1)}\frac{d^n}{dx^n}[x^{q+n-1}(1-x)^{p+n-q}].$$

Since $\int_0^1 p(x).G_m(x)\,G_n(x)\,dx=0$

(may be proved easily).

**§ 9·15. Bessel's Inequality and Completeness Relation.**

**[Meerut 76, 86]**

Let $\psi_1, \psi_2 \ldots$ be an orthonormal system and let $f$ be any function. Then the numbers $C_\nu$, $\nu=1, 2, \ldots$ such that

$$C\nu=(f,\psi_\nu)=\int f\,\psi_\nu\,dx$$

are called the **expansion coefficients of components of** $f$ w.r.t. the given orthonormal system.

Obviously $\displaystyle\int\left(f=\sum_{\nu=1}^{n} C_\nu\psi_\nu\right)^2 dx\geqslant 0.$ ...(ii)

By writing out the square and integrating term by term, we

$$0\leqslant\int f^2\,dx-2\sum_{\nu=1}^{n} C_\nu\int f.\phi_\nu\,dx+\sum_{\nu=1}^{n} C^2{}_\nu$$

or $\qquad 0\leqslant(Nf)^2-2\sum_{\nu=1}^{n} C^2\nu+\sum_{\nu=1}^{n} C^2\nu$

[$Nf$ means norm of $f$]

or $\qquad 0\leqslant(Nf)^2-\sum_{\nu=1}^{n} C^2\nu$

or $\qquad \sum_{\nu=1}^{n} C^2{}_\nu\leqslant(\mathrm{Nf})^2.$ ...(iii)

Since the number on right is independent of $n$, it follows that

$$\sum_{\nu=1}^{\infty} C^2_\nu < (Nf)^2.$$

**This fundamental inequality is known as "Bessel's inequality"** and is true for every orthonormal system. It proves that the sum of the squares of the expansion coefficients always converges.

For systems of functions with complex values of correspon-ding relation

$$\sum_{\nu=1}^{n} | C_\nu |^2 \leqslant (Nf)^2=(f, \bar{f}) \qquad \ldots(\text{iv})$$

holds, where $C_\nu$ is the expansion coefficient $C_\nu=(\bar{f}, \psi_\nu)$.

This relation may be obtained from the inequality

$$\int | f(x) - \sum_{\nu=1}^{n} C_\nu\psi_\nu |^2 \, dx=(Nf)^2 - \sum_{\nu=1}^{n} | C_\nu |^2 \geqslant 0.$$

The **significance** of the integral in (ii) is that it occurs in the **problem of approximating** the given function $f(x)$ by a linear com-bination $\sum_{\nu=1}^{n} \lambda_\nu\psi_\nu$ with $\lambda_\nu$ as constant coefficient and fixed $n$, in such a way that the **mean square** error ..

$$M=\int (f - \sum_{\nu=1}^{n} \lambda_\nu\psi_\nu)^2 \, dx$$

is as small as possible.

[Since, by simple manipulation of the integral, we obtain

$$M=\int (f - \sum_{\nu=1}^{n} \lambda_\nu\psi_\nu)^2 \, dx$$

$$=\int f^2 \, dx+ \sum_{\nu=1}^{n} (\lambda_\nu - C_\nu)^2 - \sum_{\nu=1}^{\infty} C_\nu^2]$$

which follows immediately that $M$ is least for $\lambda_\nu=C$].

An approximation of this type is known as an approximation by the method of least squares, or an **approximation "in the mean"**.

If, for a given orthonormal system $\psi_1, \psi_2, \ldots$, any piecepise continouous function $f$, can be approximated in the mean to any desired degree of accuracy by choosing $n$ large enough, *i.e.*, if $n$ may be so chosen that the mean square error

$$\int (f - \sum_{\nu=1}^{n} C_\nu \psi_\nu)^2 \, dx$$

is less than a given arbitrary small positive number, then the system of functions $\psi_1, \psi_2, \ldots\ldots$, is said to be **Complete.**

For a complete or orthonormal system of functions Bessel's inequality becomes an equality for every function $f$

*i.e.*
$$\sum_{\nu=1}^{n} C_\nu^2 = (N f)^2$$

or
$$\sum_{\nu=1}^{\infty} (f, \psi_\nu)^2 = \| f \|^2.$$

The relation is know as the "**Completeness relation**" or **Persval's equation.**

**§ 9·16. Definition.** **[Meerut 71, 75]**

**Closed Set.** The set $\{\phi_n\}$ is **closed** in the sense of mean convergence if for each function $f$ of the function space.

$$\sum_{n=1}^{\infty} (f, \phi_n)^2 = \| f \|^2$$

**Complete Set.** An orthonormal set $\{\phi_n\}$ is **complete** in the function space if there is no function in that space, with positive norm, which is orthogonal to each of the functions $\phi_n$.

**§ 9·17. Theorem. If an orthonormal set $\{\phi_n(x)\}$ is closed it is complete.** **[Meerut 75, 82 (P)]**

**Proof.** If an orthonormal set $\{\phi_n(x)\}$ is closed then for each function $f$ of the function space

$$\sum_{n=1}^{\infty} (f, \phi_n)^2 = \| f \|^2 \qquad \ldots(i)$$

Now let us suppose a function $\psi$ in the space which is orthogonal to each function $\phi_n(x)$ of the closed orthonormal set $\{\phi_n(x)\}$ such that

$$\| \phi \| \neq 0.$$

$$\therefore \quad (f, \phi_n) = 0$$

$\therefore$ from (i), we have $\| f \| = 0$

which is a contradiction

Therefore there is no function in space, with positive norm which is orthogonal to each of the function $\phi_n(x)$.

Hence the orthonormal set $\{\phi_n(x)\}$ is closed. **Proved.**

## Exercise on chapter IX

1. Show that the function sin $x$, sin $2x$, sin $3x$,..., are orthogonal on the interval $(0, \pi)$. [Meerut 73]

2. With the help of 1, $x$, $x^2$ construct three functions $g_0$; $g_1$ and $g_2$ which are orthogonal.
   (i) Over $0 \leqslant x \leqslant \infty$ with respect to $e^{-x}$. [Meerut 74]
   (ii) Over $-\infty < x < \infty$ with respect to $e^{-x^2}$. [Meerut 74 (S)]

3. In the following cases, show that the given set is orthogonal on the given interval, and determine the corresponding orthonormal set.
   (a) $\sin \pi x$, $\sin 2\pi x$ $\sin 3\pi x$... $\quad -1 \leqslant x \leqslant 1$.
   (b) $1$, $\cos 2x$, $\cos 4x$, $\cos 6x$,... $\quad 0 \leqslant x \leqslant \pi$
   (c) $1, \cos \frac{2\pi n}{T} x$, $\sin \frac{2n\pi}{T} x$, $(n=1, 2 \ldots)$ $-\frac{T}{2} \leqslant x \leqslant \frac{T}{2}$.
   (d) $\{\cos nx\}$, $n=0, 1, 2, \ldots$ $\quad 0 \leqslant x \leqslant \pi$.
   [Meerut 77 (S), 84 (P)

4. Show that the functions
   $$1-x,\ 1-2x+\tfrac{1}{2}x^2 \text{ and } 1-3x+(3/2)x^2-(1/6)x^3$$
   are orthogonal with respect to $e^{-x}$ on $0 \leqslant x \leqslant \infty$. Determine the corresponding orthogonal functions. [Meerut 81 (S)]

5. Show that the function $f_1(x)=4$ and $f_2(x)=x^3$ are orthogonal on the interval $(-2, 2)$ and determine the constants $A$ and $B$ so that the function $f_3(x)=1+Ax+Bx^2$ is orthogonal to both $f_1$ and $f_2$.

6. Find all eigen values and eigen functions of the Strum-Liouville problem $\quad X''+\lambda X'=0$.
   $$X'(-\pi)=0,\ X'(\pi)=0.$$ [Meerut 88]

7. Find all eigen values and eigen functions of the Strum-Liouville problem
   $X''+\lambda X=0$, $X(0)=0$, $X=(1)-x'(1)=0$. [Meerut 83 (P)]

8. Find all eigen values and eigen functions of the Strum-Liouville's problem
   $(x^2X')'+\lambda xX=0$, $X(1)=0$, $X(c)=0$. [Meerut 84 (P)]

9. Find the characteristic functions of the following case of the Strum-Liouville's problem;—
   $$X''+\lambda X=0,\ X(0)=0,\ X'(L)=0$$
   [Meerut 76, 77 (S)]

10. State Strum-Liouville problem and discuss orthogonality of the functions and the nature of eigen values. [Meerut 83]

11. For the eigen value problem given below, obtain the set of orthogonal eigen function in the interval $(0, 2c)$.

$$X''+\lambda X=0,\ X(0)=X(2c),\ X'(0)=X'(2c)$$ [Meernt 83, 87]

12 Show that the functions $f_1(x)=4$ and $f_2(x)=x^3$ are orthogonal in the intenal $(-2, 2)$ and determine the constants A and B so that the functions $f_3(x)=1+Ax+Bx^2$ is orthogonal to both $f_1$ and $f_2$. [Meerut 85]

13. If $L_n(x)=e^x \dfrac{d^n}{dx^n}(x^n e^{-x})$, then show that

$$\psi_n(x)=\frac{1}{n!}e^{-x/2}L_n(x)$$

forms an orthogonal system in $(0, \infty)$ [Meerut 88]

[Hnit. Sce § 9·12]

## ANSWERS

3. (a) $\sin \pi x, \sin 2\pi x, \ldots$

(b) $\dfrac{1}{\sqrt{\pi}}, \sqrt{\left(\dfrac{2}{\pi}\right)} \cos 2x, \sqrt{\left(\dfrac{2}{\pi}\right)} \cos 4x, \ldots$

(c) $\dfrac{1}{\sqrt{T}}, \sqrt{\dfrac{2}{\pi}} \cos \dfrac{2n\pi x}{T}, \sqrt{\left(\dfrac{2}{T}\right)} \sin \dfrac{2n\pi x}{T} \ldots$

$n=1, 2, \ldots$

(d) $\dfrac{1}{\sqrt{\pi}} \sqrt{\left(\dfrac{2}{\pi}\right)} \cos nx, n=1, 2, \ldots$

4. $1-x$, $1-2x+\frac{1}{2}x^2$, and $1-3x+\frac{3}{2}x^2-\frac{1}{6}x^3$

6. $X(x)=\cos \dfrac{n(\pi+x)}{2}$, $n=0, 1, 2, \ldots$

7. $X_0=x$, $X_n=\sin \lambda_n x$, $n=1, 2, \ldots$

8. $X_n=x^{-1} \sin (n\pi \log x)$, $n=1, 2, \ldots, p(x)=x$.

9. $X(x)=\dfrac{\sin (2n-1)\pi x}{2L}$, $n=1, 2.$

# SPECIAL FUNCTIONS

## PART II

*Chapters.*

# Elliptic Functions

§ 1·1. **Periodic Function** [Kanpur 84], *A function $f(z)$ is said to be a periodic function if there exists a non-zero constant $T$ such that.*

$$F(z) = F(z+T) = F(z+2T) = F(z+3T) = \ldots \text{ etc.}$$

for all values of z. This constant $T$ is called the period of the function $f(z)$. Clearly if $T$ is a period of $F(z)$ then $nT$ (where $n$ is a positive or negative integer) is also a period.

**Fundamental Period.** [Kanpur 84], The period $T$ is called the fundamental period of $f(z)$ if no submultiple of it is a period of the function $f(z)$.

**Simple-periodic function.** A periodic function which has only one fundamental period is said to be simply-periodic.

**Multiply-periodic function.** A periodic function which has more than one fundamental period is said to be multiply-periodic.

Note. If $2\omega_1$ and $2\omega_2$ is a pair of primitive periods of a doubly periodic function $f(z)$ then the periods of $f(z)$ are of the form $2m\omega_1 + 2n\omega_2$, where $m, n$ are integers.

§ 1·2. **Elliptic Function.** [Kanpur 83; 84], *A doubly-periodic analytic function $f(z)$ is said to be an* **elliptic function** *if its only possible singular points in the finite part of the plane are poles.*

*OR*

*All the meromorphic functions which have two distinct periods are called* **elliptic fuctions.**

**Primitive period parallelogram, Mess and cell (Definitions).** **[Kanpur 83]**

Let $f(z)$ be an elliptic function with $2\omega_1$ and $2\omega_2$ as a pair of primitive periods.

Supposing that imaginary part of $\frac{\omega_2}{\omega_1}$ which is not zero, to be positive, we see that the points $0, 2\omega_1, 2\omega_1+2\omega_2, 2\omega_2$ taken in order are the vertices of a parallelogram described in the positive sense. This is called the **primitive period-parallelogram** of the

elliptic function $f(z)$. There are the unlimited number of such primitive period parallelograms. Now the only periods of $f(z)$ (doubly periodic function) are of the form $2m\omega_1+2n\omega_2$, where $m$ and $n$ are integers. Therefore the vertices are the only points within or on a primitive period-parallelogram whose affixes are *₁ (periods. In the Argand plane the points) of affix $2m\omega_1+2n\omega_2$ where $m=0, \pm 1, \pm 2, \ldots, n=0, \pm 1, \pm 2, \ldots$ are denoted by $\Omega_{m,n}$

i.e. $\Omega_{m,n}=2m\omega_1+2n\omega_2$.

Thus the four points $\Omega_{m,n}, \Omega_{m+1,n}, \Omega_{m+1,n+1}, \Omega_{m,n+1}$ are the vertices of a parallelogram. This parallelogram may be obtained from the primitive period-parallelogram by a translation without rotation. This parallelogram with vertices $\Omega_{m,n}, \Omega_{m+1,n}, \Omega_{m+1,n+1}, \Omega_{m,n+1}$ is called a **period parallelogram or a mesh.** The Argand plane may be covered by this system of non overlapping meshes. In each mesh (within and on the boundary) there are only a finite number of poles and zeros (see § 1·3), and hence we can translate the mesh without rotation until no pole or zero lies on its boundary. **The parallelagram thus obtained is called a cell.**

The set of poles (or zeros) in a given cell is called an **irreducible set.**

**§ 1·3. Order of an elliptic function.** *The order of an elliptic function is defined as the number of its poles in a cell.*

**1·4. Properties of an Elliptic Function.**

**(1) The behaviour of an elliptic function is completely determined by values in a primitive period-parallelogram.**

**Proof.** The points $z+\Omega_{m,n}$ and $z=\Omega_{p,q}$ lie in different meshes. These two points will be coincident points when one mesh is translated untill it coincides with the second mesh. We say that these two points are the congruent points. Now $\Omega_{m,n}-\Omega_{p,q}$ is a period of the function $f(z)$ which follows that $f(z)$ takes the same values at the congruent points $z=\Omega_{m,n}$ and $z=\Omega_{p,q}$. Thus the function $f(z)$ takes the same value at every one of a set congruent points. Hence the behaviour of an elliptic function is completely determined by its values in a primitive period-parallelogram.

**(2) An elliptic function f (x), must posses poles.**

**Proof.** Let $f(z)$ be an elliptic function which is regular in the primitive period-parallelogram (since only singular points of an

*₁ (periods in the Argand plane the points)

elliptic function are poles). Thus $|f(x)| < k$ for all points in the primitive period-parallelogram, where $k$ is a finite constant. But the values taken by $f(z)$ in the primitive period-parallelogram are repeated in every mess [followed by (1)]. Thus $f(z)$ is an integral function satisfying the inequality $|f(z)| < k$ for all values of $z$. Hence by Lioville's theorm (see complex varibles) the function $f(z)$ must be constant. Hence an elliptic function $f(z)$ must possess poles.

**Note.** For the elliptic function $f(z)$ one of the periods can always be assumed as real while the other will in general be complex.

(3) **An elliptic function f (z). with no poles in a cell is merely a constant.** **[Kanpur 71]**

**Proof.** See (2).

(4) **An elliptic function has only a finite number of poles in any mesh.**

**Proof.** If an elliptic function $f(z)$ has an infinite number of poles then they will possess a limiting point. But a limiting point of poles is an essential singularity which is impossible since the only singular points of an elliptic function in the finite part of the plane are poles. Thus the elliptic function $f(z)$ can not have an infinite number of poles. Hence an elliptic function has only a finite number of poles in any mesh.

(5) **An elliptic function has only a finite number of zeros in any mesh.**

**Proof.** Prove similarly as property (4),

(6) **The sum of the residues of an elliptic function f (z) at its poles in any cell is zero.** **[Kanpur 71]**

**Proof.** Let $c$ be a cell. The function $f(z)$ (elliptic function) is regular within and on the closed contour $C$, save for a finite number of poles, within $C$; by Cauchy's theorem of residues, we have

$$\int_C f(z)\, dz = 2\pi i \Sigma \mathrm{R}^+$$

or $\Sigma \mathrm{R}^+$ (sum of residues of $f(z)$ at its poles within $C$)

$$= \frac{1}{2\pi i} \int_C f(z)\, dz.$$

But $\int_C f(z)\, dz = 0$; since the integral along opposite side of $C$ being of equal and opposite signs, cancel ($f(z)$ is a periodic

function and opposite sides are traversed in opposite senses)

$$\therefore \quad \Sigma R^{+}=0$$

*i.e.* the sum of the residues of an elliptic function at its poles in any cell is zero.

**(7) The order of an elliptic function is at least two.**

**Proof.** Recall that the order of an elliptic function is the number of its poles in any cell (each pole is counted according to its multiplicity). If the order of an elliptic function $f(z)$ is one, $f(z)$ has only one irriducible (*i.e.*, pole of order one) in the cell. According to property (6) the residue of the function $f(z)$ at this pole is zero, which is impossibe since one irreducible pole always has a residue different from zero. Hence the order of an elliptic function $f(z)$ is at least two.

**(8) An elliptic function of order m has m zeros in each cell (each zero is counted according to its multiplicity).**

**Proof.** Let $f(z)$ be elliptic function of order $m$.

Let $n$ be the number of zeroes of the function $f(z)$ in a cell. Then from complex variable, we have

$$n-m=\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$$

or $n-m=$ sum of residues of $\dfrac{f'(z)}{f(z)}$ at its poles in the cell.

But $f(z)$ is an elliptic function therefore $f'(z)$ is also an elliptic function with the same period as $f(z)$.

So $\dfrac{f'(z)}{f(z)}$ is an elliptic function and therefore from property (6) the sum of residues of $\dfrac{f'(z)}{f(z)}$ at its poles in the cell is zero.

Hence $n-m=0$ or $n=m$.

Hence an elliptic function of order $m$ has $m$ zeros in each cell.

**§ 1·5. Weierstrass's functions.**

**Weierstrass's Sigma functions.** Weirstrass's Sigma function is given by

$$\sigma(z/\omega_1, \omega_2)=z\prod_{m,n=-\infty}^{\infty}\left[\left(1-\frac{z}{\Omega_{m\cdot n}}\right).\exp.\left(\frac{z}{\Omega_{m\cdot n}}+\frac{z^2}{2\Omega^2_{m\cdot n}}\right)\right]$$

where multiplication is extended over all positive and negative integral and zero values of $m$ and $n$, save $m=n=0$.

$\sigma(z/\omega_1, \omega_2)$ is an integral function of order 2, with simple zeroes at the points $\Omega_{m,n}$. This function is also denoted by $\sigma(z)$.

**§ 1·6. Properties of Weierstrass's Sigma function.**

**1. $\sigma(z)$ is an odd function of z.** In the product of the factors in $\sigma(z)$ the factors may be arranged in pairs, such as

$$\left[\left(1-\frac{z}{\Omega_{m,n}}\right).\exp.\left(\frac{z}{\Omega_{m,n}}+\frac{z}{2\Omega^2_{m,n}}\right)\right]$$
$$\times\left[\left(1+\frac{z}{\Omega_{m,n}}\right).\exp.\left(-\frac{z}{\Omega_{m,n}}+\frac{z^2}{2\Omega^2_{m,n}}\right)\right]$$

(the second factor is derived from the first by replacing $m$ and $n$ by $-m$ and $-n$ respectively and using

$\Omega_{-m,-n}=2(-m)\,\omega_1+2(-n),\ \omega_2=\ (2m\omega_1+2n\omega_2)=-\Omega_{m,n}$.

Now when $z$ is replaced by $-z$ the two factors interchange keeping their product unchanged

*i.e.* $\qquad \sigma(-z)=-\sigma(z)$.

Hence $\sigma(z)$ is an odd function of $z$.

**2. $\sigma(z)$ is not an elliptic function.**

**§ 1·6. Weierstrass's Zeta function.** Weierstrass's zeta function is defined by the equation

$$\zeta(z)=\frac{d}{dz}\log\sigma(z)$$

*i.e.* $$\zeta(z)=\sum_{m,\,n=-\infty}^{\infty}\left[\frac{1}{z-\Omega_{m,n}}+\frac{1}{\Omega_{m,n}}+\frac{z}{\Omega^2_{m,n}}\right].$$

where summation is extended over all positive and negative integral and zero values of $m$ and $n$, save $m=n=0$.

$\zeta(z)$ is an analytic function with simple poles of residue 1 at each of the points $\Omega_{m,n}$

**§ 1·7, Properties of $\zeta(z)$.**

**1. $\zeta(z)$ is an odd function of z.**

$\zeta(z)$ may also be written as

$$\zeta(z)=\frac{1}{z}\sum_{m,n=-\infty}^{\infty}\left[\left(\frac{1}{z-\Omega_{m,n}}+\frac{1}{\Omega_{m,n}}+\frac{z}{\Omega^2_{m,n}}\right)\right.$$
$$\left.+\left\{\frac{1}{z+\Omega_{m,n}}-\frac{1}{\Omega_{m,n}}+\frac{z}{\Omega^2_{m,n}}\right)\right]$$

(the second term in summation is derived from first by replacing $m, n$ by $-m$ and $-n$)

where the summation is extended over positive integral and zero values of $m, n$, save $m=n=0$.

Replacing $z$ by $-z$, we have

$$\zeta(-z)=-\frac{1}{z}+\sum_{m,n=-\infty}^{\infty}\left[\left(\frac{1}{-z-\Omega_{m,n}}+\frac{1}{\Omega_{m,n}}-\frac{z}{\Omega^2_{m,n}}\right)\right]$$
$$+\left[\left(-\frac{1}{-z+\Omega_{m,n}}-\frac{1}{\Omega_{m,n}}-\frac{z}{\Omega^2_{m,n}}\right)\right]$$
$$=-\zeta(z).$$

Hence $\zeta(z)$ is an odd function of $z$.

**2. $\zeta$ (z) is not elliptic function.** $\zeta(z)$ is an analytic function with simple poles of residue 1 at each of the points $\Omega_{m,n}$.

$\therefore$ the sum of residues $\zeta(z)$ at all its poles is not zero but the sum of residues of an elliptic function at all its simple poles is zero.

Hence $\zeta$ (z) is not elliptic function.

**§ 1·8. Weierstrass's elliptic function.** **[Kanpur 71, 72, 83, 84, 85, 87]**

**Weierstrass's elliptic function** *or simply.* **Weierstrass function** *p (z) is defined by the equation*

$$p(z)=-\frac{d}{dz}\zeta(z).$$

*i.e.*
$$p(z)=\frac{1}{z^2}+\sum_{m,n=-\infty}^{\infty}\left[\frac{1}{(z-\Omega_{m,n})^2}-\frac{1}{\Omega^2_{m,n}}\right]$$

where summation is extended **over all** positive and negative integral and zero values of $m$ and $n$, **save** $m=n=0$.

$p(z)$ is an analytic function whose only singularities are double poles of residue zero at each of the points $\Omega_{m,n}$.

**§ 1·9. Properties of p (z).**

**1. p (z) is an even function of z.** **[Kanpur 72]**

$p(z)$ may also be written as

$$p(z)=\frac{1}{z^2}+\sum_{m,n=-\infty}^{\infty}\left[\left\{\frac{1}{(z-\Omega_{m,n})^2}-\frac{1}{\Omega^2_{m,n}}\right\}+\left\{\frac{1}{(z+\Omega_{m,n})^2}-\frac{1}{\Omega^2_{m,n}}\right\}\right]$$

(the second term in summation is derived from first by replacing $m$ and $n$ by $-m$, $-n$ repeciively and using the fact that

$$\Omega_{-m,-n}=2(-m)\,\omega_1+2(-n)\,\omega_2=-(2m\omega_1+2n\omega_2)=-\Omega_{m,n}$$

where summation is extended over all positive integral and zero values of $m$ and $n$, save $m=n=0$.

Replacing $z$ by $-z$ we have

$$p(-z)=\frac{1}{z^2}+\sum_{m,n=-\infty}^{\infty}\left[\left\{\frac{1}{(-z-\Omega_{m,n})^2}-\frac{1}{\Omega^2_{m,n}}\right\}+\left\{\frac{1}{(-z+\Omega_{m,n})^2}-\frac{1}{\Omega^2_{m,n}}\right\}\right]$$

$$=p(z).$$

Hence $p(z)$ is an even function of $z$.

2. **$p$ (z) is an elliptic function.** **[Kanpur 72, 83]**

We have $p(z)=\frac{1}{z^2}+\sum_{m,n=-\infty}^{\infty}\left[\frac{1}{(z-\Omega_{m,n})^2}-\frac{1}{\Omega^2_{m,n}}\right]$

$$\therefore\quad p'(z)=-\frac{2}{z^3}-\sum_{m,n=-\infty}^{\infty}\frac{2}{(z-\Omega_{m,n})^3}$$

where the summation is extended over all positive and negative integral and zero values of $m$ and $n$, save $m=n=0$.

We may also write that

$$p'(z)=-\sum_{m,n=-\infty}^{\infty}\frac{2}{(z-\Omega_{m,n})^3}$$

where the summation is extended over all positive and negative integral and zero values of $m$ and $n$.

Now $p'(z+2\omega_1)=-\sum_{m,n=-\infty}^{\infty}\frac{2}{(2+2\omega_1-\Omega_{m,n})^3}$

Since $2\omega_1-\Omega_{m,n}=2\omega_1-(2m\omega_1+2n\omega_2)$

$=-\{2(m-1)\,\omega_1+2n\omega_2\}$

$=-\Omega_{m-1,n}$

$$=\sum_{m,n=-\infty}^{\infty}\frac{1}{(z-\Omega_{m-1,n})^3}$$

$$=-\sum_{m,n=-\infty}^{\infty}\frac{2}{(z-\Omega_{m,n})^3}$$

Since the set of points $\Omega_{m-1,n}$, is the same as the set of points $\Omega_{m,n}$ for all positive and negative integral and zero values of $m$ and $n$.

$$\therefore \quad p'(z+2\omega_1)=p'(z)$$

*i.e.* $p'(z)$ is a periodic function with period $2\omega_1$. Similarly we can prove that $p'(z)$ is a periodic function with period $2\omega_1$.

Now $p'(z)$ has poles of order 3 at each of the points $\Omega_{m,n}$ where the residue is zero.

Thus $p'(z)$ is a doubly periodic function for which the sum of residues at all its poles is zero. Also $p'(z)$ is analytic function as $p(z)$ is analytic function.

$$\therefore \quad p'(z) \text{ is an elliptic function}$$

Now integrating $\qquad p'(z+2\omega_1)=p'(z)$

we have $\qquad p(z+2\omega_1)=p(z)+c$

Now putting $z=-\omega_1$ we have

$$p(\omega_1)=p(-\omega_1)+c$$

or $\qquad c=p(\omega_1)-p(-\omega_1)$

$$=p(\omega_1)-p(\omega_1)=0$$

(Since $p(z)$ is an even function of $z$)

$$\therefore \quad p(z+2\omega_1)=p(z)$$

Similarly $\qquad p(z+2\omega_2)=p(z)$.

$\therefore$ $2\omega_1$, $2\omega_2$ are also primitive periods of $p(z)$.

Thus $p(z)$ is doubly periodic analytic function whose only singularities are poles.

Hence $p(z)$ is an elliptic function.

**3. p′ (z) is an odd function of z.** **[Kanpur 1972, 84]**

We have,

$$p(z)=\frac{1}{z^2}+\sum_{m,n=-\infty}^{\infty}\left[\frac{1}{(z-\Omega_{m,n})^2}-\frac{1}{(\Omega^2_{m,n})}\right] \qquad \ldots(1)$$

$m=n=0$ is omitted.

The series in (1) is uniformly convergent and so the term by term differentiation is valid. Therefore differentiating (1), we get

$$p'(z)=\sum_{m,n=-\infty}^{\infty}\left[\frac{-2}{(z-\Omega_{m,n})^3}\right] \qquad \text{Note.} \qquad \ldots(2)$$

the term for which $m=n=0$ is omitted

Replacing $z$ by $-z$ in (2), we get

$$p'(-z)=\sum_{m,n=-\infty}^{\infty}\left[\frac{-2}{(-z-\Omega_{m,n})^3}\right] \qquad \ldots(3)$$

$$= \sum_{m, n=-\infty}^{\infty} \frac{-2}{(-z-\Omega_{-m,-n})^3}\Bigg]$$

obtained by replacing $m$ by $-m$ and $n$ by $-n$ in (3)

$$= \sum_{m, n=-\infty}^{\infty} \left[\frac{-2}{(-z+\Omega_{m,n})^3}\right] \quad (\because \ \Omega_{-m,-n}=-\Omega_{m,n})$$

$$= \sum_{m, n=-\infty}^{\infty} \left[\frac{2}{(z-\Omega_{m,n})^3}\right]$$

$$= -p'(z) \qquad \text{from (2)}$$

*i.e.* $\quad p'(-z)=-p'(z).$

Hence $p'(z)$ is an odd function of $z$.

**4. p′ (z) is an elliptic function.** **[Kanpur 1972]**

Proceeding as in 3, we have

$$p'(z)=\sum_{m, n=-\infty}^{\infty} \left[\frac{-2}{(z-\Omega_{m,n})^3}\right] \qquad \text{...(1)}$$

Here in order to prove that $p'(z)$ is an elliptic function, it is sufficient to prove that $p'(z)$ have two distinct periods say $2\omega_1$ and $2\omega_2$.

From (1), we have

$$p'(z+2\omega_1)=\sum_{m, n=-\infty}^{\infty} \left[\frac{-2}{(z+2\omega_1-\Omega_{m,n})^2}\right] \qquad \text{...(2)}$$

Now $z+2\omega_1-\Omega_{m,n}=z+2\omega_1-(2m\omega_1+2n\omega_2)$

$$=z-\{2(m-1)\,\omega_1+2n\omega_2\}$$

$$=z-\Omega_{m-1,n}.$$

$\therefore$ From (2), we get

$$p'(z+2\omega_1)=\sum_{m, n=-\infty}^{\infty} \left[\frac{-2}{(z-\Omega_{m-1,n})^2}\right]$$

Replacing $m$ by $m-1$, we get

$$p'(z+2\omega_1)=\sum_{m, n=-\infty}^{\infty} \frac{-2}{(z-\Omega_{m,n})^3}=p'(z). \qquad \text{...(3)}$$

Proceeding similarly, we get

$$p'(z+2\omega_2)=p'(z) \qquad \text{...(4)}$$

From (3) and (4) it follows that $p'(z)$ have two distinct periods $2\omega_1$ and $2\omega_2$. Hence $p'(z)$ is an elliptic function.

**§ 1·10 An algebraic relation connecting two elliptic functions.**

**Theorem.** *If two elliptic functions have a pair of common periods whose ratio is not real, they are connected by an algebraic relation.*

**Proof.** The proof is out of scope of the present volume.

**§ 1·11. The differential equation satisfied by the Weierstrass's elliptic function p (z).** [Kanpur 70, 71, 86]

We have already seen that if $2\omega_1$ and $2\omega_2$ are the primitive periods or the Weierstrass's elliptic function $p(z)$ then these are also the primitive periods of $p'(z)$ which is also an elliptic function. Thus $p(z)$ and $p'(z)$ are two elliptic functions having a pair of common periods whose ratio is not real. So according to theorem § 1·10 their exists a relation connecting them ($p(z)$ and $p'(z)$). This relation is actually a differential equation satisfied by $p(z)$.

Now the function $p(z)-\dfrac{1}{z^2}$ is regular in a neighbourhood of the origin and hence can be expanded in power series *i.e.*

$$p(z)-\frac{1}{z^2}=\sum_{n=0}^{\infty} a_n z^n. \qquad \text{...(i)}$$

From Taylor's theorem it can be seen easily that

$$a_1=a_3=a_5=\ldots=0 \text{ (each)}$$

and $a_0=0,\quad a_2=3\sum_{m,n}\Omega^{-4}\quad a_4=5\sum_{m,n}\Omega^{-6}$ etc.

$\therefore$ from (i), we have

$$p(z)=\frac{1}{z^2}+a_2z^2+a_4z^4+o(z^4). \qquad \text{...(ii)}$$

where $o(z^n)$ denotes a function regular in a neighbourhood of the origin and has a zero of order $n$ at the origin.

From (ii), we have

$$p'(z)=-2z^{-3}+2a_2z+4a_4z^3+o(z^5). \qquad \text{...(iii)}$$

Now $p'^2(z)=4z^{-6}-8a_2z^{-2}-16a_4+o(z^2)$ ...(iv)

and $p^3(z)=z^{-6}+3a_2z^{-2}+3a_4+o(z^2).$ ...(v)

Subtracting 4 times of equation (v) from (iv), we have

$$p'^2(z)-4p^3(z)=-20a_2z^{-2}-28a_4+o(z^2) \qquad \text{...(vi)}$$

But from (ii), we have

$$p(z)=z^{-2}+o(z^2)$$

$$\therefore\ z^{-2}=p(z)-o(z^2)$$

Substituting in (vi), we have

$$p'^2(z)-4p^3(z)=-20a_2\,p(z)-28a_4+o(z^2)$$

or $$p'^2(z)-4p^3(z)+20a_2\,p(z)+28a_4=o(z^2)$$

i.e $$\phi(z)=p'^2(z)-4p^3(z)+20a_2\,p(z)+28a_4 \quad \text{...(vii)}$$

is a function regular in a neighbourhood of the origin and has a zero of order 2 at the origin.

Now $\phi(z)$ is in elliptic function with periods $2\omega_1$ and $2\omega_2$. hence regular in a neighourhood of each of the points $\Omega_{m,n}$. But the points $\Omega_{m,n}$ are the only possible singularities of $\phi(z)$. Hence $\phi(z)$ is an elliptic function with no singularities, and so is constant *i. e.* $\phi(z)=A$ (constand).

Since the function $\phi(z)$ has a double zero at the origin

$$\therefore\ \phi(0)=0 \text{ and } \phi'(0)=0$$

which gives $A=0$.

$$\therefore\ \phi(z)=p'^2(z)-4p^3(z)+20a_2\,p(z)+28a_4=0$$

or $$p'^2(z)=4p^3(z)-20a_2\,p(z)-28a_4 \quad \text{...(viii)}$$

where $$a_2=3\Sigma\,\Omega_{m,n}^{-4} \text{ and } a_4=5\Sigma\,\Omega_{m,n}^{-6}$$

Now if $p(z)=\omega$ then $p'(z)=\dfrac{d\omega}{dz}$

$\therefore$ from (viii), we have

$$\left(\frac{d\omega}{dz}\right)^2=4\omega^3=20a_2\omega-28a_2. \quad \text{...(ix)}$$

Thus (ix) is **the differential equation satisfied by the Weierstrass's elliptic function p (x).**

Differential equation (ix) may also be written as

$$\left(\frac{d\omega}{dz}\right)^2=4\omega^3-60\,(\Sigma\,\Omega_{m,n}^{-4})\,\omega-140\,(\Sigma\,\Omega_{m,n}^{-6})$$

or $$\left(\frac{d\omega}{dz}\right)^2=4\omega^3-g_2\omega-g_3$$

**[Kanpur 87]**

where $g_2=60\ \Sigma\,\Omega_{m,n}^{-4}$ and $g^3=140\ \Sigma\,\Omega_{m,n}^{-6}$

are called the invariants of $p(z)$.

§ **1·12. The three roots $e_1$, $e_2$, $e_3$, of the equation**

$$4\omega^2-g_2\omega-g_3=0$$

**are all distinct**

We have the equation

$$4\omega^3-g_2\omega-g_3=0 \qquad \text{...(i)}$$

where $\omega=p(z)$ and $g_2, g_3$ are invariants of $p(z)$.

Let $e_1, e_2, e_3$ be the values taken by the Weierstrass's elliptic function $p(z)$ at the points where its derivative vanishes.

Since $p'(z)$ is an odd periodic function

$$\therefore\ p'(\omega_1)=p'(\omega_1-2\omega_1)=p'(-\omega_1)$$
$$=-p'(\omega_1)$$

or $$2p'(\omega_1)=0$$

$$\therefore\ p'(\omega_1)=0.$$

Similarly $$p'(\omega_2)=0.$$

Since $p'(z)$ is an elliptic function of order 3 with a pole of order 3 at each of the points $\Omega_{m\,n}$. [See § 1·4 (2)]

$\therefore$ The sum of the affixes of irreducible zeros of $p'(z)$ is a period.

Also $$\omega_1+\omega_2+\omega_3=0.$$

$\therefore$ the points $\omega_1, \omega_2, \omega_3$ form a set of irreducible zeros of $p'(z)$.

$$\therefore\ e_1=p(\omega_1),\ e_2=p(\omega_2),\ e_3=p(\omega_3)$$

are the roots of equation (i).

**To prove that $e_1, e_2, e_3$ are distinct.** We have $p(z)-e_1$ an elliptic function of order 2 with a double zero at $z=\omega_1$.

$\therefore$ $p(z)-e_1$ cannot vanish at any other point in the primitive period-parallelogram.

$$\therefore\ e_1\neq e_2 \text{ and } e_1\neq e_3.$$

Similarly $e_2\neq e_3$.

Hence $e_1, e_2, e_3$ the three roots of the equation

$$5\omega^2-g_2\omega-g_3=0$$

are all distinct.

**§ 1·13. The pseudo-periodicity of $\zeta(z)$ and $\sigma(z)$.**

Since $p(z)$ is a doubly periodic function with $2\omega_1$ and $2\omega_2$ as its primitive periods.

$$\therefore\ p(z+2\omega_1)=p(z)$$

or $$\frac{d}{dz}\zeta(z+2\omega_1)=\frac{d}{dz}\zeta(z).$$

Integrating, we have

$$\zeta(z+2\omega_1)=\zeta(z)+2\eta_1$$

whese $2\eta_1$ is a constant of integration.

Putting $z=-\omega_1$, we have

$$\zeta(-\omega_1+2\omega_1)=\zeta(-\omega_1)+2\eta_1$$

or
$$\zeta(\omega_1)=-\zeta(\omega_1)+2\eta_1.$$

$\because$ $\zeta(z)$ is an odd function of $z$

$$\therefore \quad 2\eta_1=2\zeta(\omega_1)$$

or
$$\eta_1=\zeta(\omega_1).$$

$$\therefore \quad \zeta(z+2\omega_1)=\zeta(z)+2\eta_1 \qquad \text{...(i)}$$

where
$$\eta_1=\zeta(\omega_1).$$

Similarly $\zeta(z+\omega_2\eta)=\zeta(z)+2\eta_2$ ...(ii)

where
$$\eta_2=\zeta(\omega_2).$$

The constants $\eta_1$ and $\eta_2$ are not both zero otherwise $\zeta(z)$ would become an elliptic function while we have already seen that $\zeta(z)$ is not an elliptic function.

From (i) and (ii) we see that the function $\zeta(z)$ is reproduced when $z$ is increased by $2\omega_2$ [periods of $p(z)$], the only difference is an additive constant.

Thus the function $\zeta(z)$ has a pesudo-periodicity.

Making use of the period $2\omega_3$, where $\omega_1+\omega_2+\omega_3=0$. the pesudo-periodicity of $\zeta(z)$ with respect to $2\omega_3$ may be expressed by the equation

$$\zeta(z+2\omega_3)=\zeta(z)+2\eta_3$$

where
$$\eta_1+\eta_2+\eta_3=0.$$

Again, we have

$$\zeta(z+2\omega_1)=\zeta(z)+2\eta_1$$

or
$$\frac{d}{dz}\log\sigma(z+2\omega_1)=\frac{d}{dz}\log\sigma(z)+2\eta_1.$$

Integrating, we have

$$\log\sigma(z+2\omega_1)=\log\sigma(z)+2\eta_1 z+\log A$$

$$\therefore \quad \sigma(z+2\omega_1)=Ae^{2\eta_1 z}\sigma(z).$$

Where $A$ is a constant of integration.

Putting $z=-\omega_1$, we have

$$\sigma(-\omega_1+2\omega_1)=Ae^{-2\eta_1\omega_1}\sigma(-\omega_1)$$

or
$$\sigma(\omega_1)=-Ae^{-2\eta_1\omega_1}\sigma(\omega_1).$$

Since $\sigma(z)$ is an odd function of $z$

or
$$A=-e^{2\eta_1\omega_1}$$

$$\therefore \quad \sigma(z+2\omega_1)=-e^{2\eta_1(z+\omega_1)}\sigma(z).$$

Similarly, we have

$$\sigma(z+2\omega_2)=-e^{2\eta_2(z+\omega_2)}\sigma(z)$$

$$\sigma(z+2\omega_3)=-e^{2\eta_3(z+\omega_3)}\sigma(z).$$

The above equations show the pseudo-periodicity of $\sigma(z)$ when $z$ is increased by a period of $p(z)$.

**§ 1·14. Jacobi's Elliptic Functions.** [Kanpur 83, 84, 86, 87]

**Definition.** *The elliptic functions with two simple poles and two simple zeros in a cell are called* **Jacobi's elliptic functions.**

**§ 1·15. Construction of Jacobi's elliptic functions.**

Let us consider the functions

$$\left.\begin{aligned} S(z)&=(e_1-e_2)^{1/2}\frac{\sigma(z)}{\sigma_2(z)},\\ C(z)&=\frac{\sigma_1(z)}{\sigma_2(z)}\\ \text{and}\quad D(z)&=\frac{\sigma_3(z)}{\sigma_2(z)}\end{aligned}\right\} \quad \ldots(\text{i})$$

where $\sigma(z)$ is Weierstrass's sigma function

and $$\sigma_r(z)=e^{-\eta_r z}\cdot\frac{\sigma(z+\omega_r)}{\sigma(\omega_r)} \qquad (r=1, 2, 3)$$

It can be easily proved that $S(z)$, $C(z)$ and $D(z)$ are all elliptic functions but they are not independent functions.

Also, we have

$$\{p(z)-e_r\}^{1/2}=e^{-\eta_r(z)}\cdot\frac{\sigma(z+\omega_r)}{\sigma(z)\,\sigma(\omega_r)} \qquad (r=1, 2, 3)$$

$$\begin{aligned}\sigma_r(z)&=e^{-\eta_r(z)}\cdot\frac{\sigma(z+\omega_r)}{\sigma(\omega_r)}\\ &=\sigma(z).\{p(z)-e_r\}^{1/2}.\end{aligned} \qquad \ldots(\text{ii})$$

$\therefore$ from (i), we have

$$\left.\begin{aligned} S(z)&=\left\{\frac{e_1-e_2}{p(z)-e_1}\right\}^{1/2}\\ C(z)&=\left\{\frac{p(z)-e_1}{p(z)-e_2}\right\}^{1/2}\\ \text{and}\quad D(z)&=\left\{\frac{p(z)-e_3}{p(z)-e_2}\right\}^{1/2}\end{aligned}\right\} \quad \ldots(\text{iii})$$

From (iii), we have

$$S'(z)=-\frac{(e_1-e_2)^{1/2}\,p'(z)}{2\{p(z)-e_2\}^{3/2}}$$

$$=\frac{(e_1-e_2)^{1/2}.[-2\{p(z)-e_1\}^{1/2}\{p(z)-e_2\}^{1/2}\{p(z)-e_3\}^{1/2}]}{2\{p(z)-e_2\}^{3/2}}$$

Since $p'(z)=-2\left[\{p(z)-e_1\}^{1/2}\{p(z)-e_2\}^{1/2}\{p(z)-e_3\}^{1/2}\right]$

or $$S'(z)=(e_1-e_2)^{1/2}.\frac{\{p(z)-e_1\}^{1/2}\{p(z)-e_3\}^{1/2}}{\{p(z)-e_2\}}$$

or $\quad S'(z)=(e_1-e_2)^{1/2}\, C(z)\, D(z)$

Similarly

$$\left.\begin{array}{ll} & C'(z)=-(e_1-e_2)^{1/2}\, D(z)\, S(z) \\ \text{and} & D'(z)=-(e_1-e_2)^{1/2}\, K^2 S(z)\, C(z) \\ \text{where} & K^2=\dfrac{e_3-e_2}{e_1-e_2}. \end{array}\right\} \qquad \text{...(iv)}$$

Now the occurrence of the factor $(e_1-e_2)^{1/2}$ in $S'(z)$, $C'(z)$ and $D'(z)$ suggests the change of the independent variable from $z$ to $u$, such that $\quad u=(e_1-e_2)^{1/2}\, z$

or $\quad z=(e_1-e_2)^{-1/2}.u$

Hence the Jacobi's elliptic function are defined by the equations

$$s_n(u)=S\{(e_1-e_2)^{-1/2}\, u\} \qquad \text{...(v)}$$

$$c_n(u)=C\{(e_1-e_2)^{-1/2}\, u\} \qquad \text{...(vi)}$$

$$d_n(u)=D\{(e_1-e_2)^{-1/2}\, u\}. \qquad \text{...(vii)}$$

Now we see that when $4\omega_1(e_1-e_2)^{1/2}$ or $4\omega_2(e_1-e_2)^{1/2}$ is added to $u$, the functions $s_n(u)$, $c_n(u)$ and $d_n(u)$ remain unaltered (since $4\omega_1$ and $4\omega_2$ are periods of $S$, $C$, $D$).

If $K=\omega_1(e_1-e_2)^{1/2}$, $iK'=\omega_2(e_1-e_2)^{1/2}$ then the number $K$, $iK'$, are called the quarte-periods of Jacobi's elliptic functions.

Thus the equations defining Jacobi's elliptic functions may also be written as (combining (i), (vi), (vii).

$$s_n(Kz/\omega_1)=\frac{k}{\omega_1}\, S(z)=\frac{K\,\sigma(z)}{\omega_1\sigma_2(z)}$$

$$c_n(KZ/\omega_1)=C(z)=\frac{\sigma_1(z)}{\sigma_2(z)}$$

$$d_n(KZ/\omega_1)=D(z)=\frac{\sigma_3(z)}{\sigma_2(z)}$$

**Note.** Jacobi's regarded $s_n(u)$, $c_n(u)$ and $d_n(u)$ as the sine, cosine and derivative of a function *am u* defined by the equation

$$u=\int_0^{amu}(1-k^2\sin^2\theta)^{-1/2}\, d\theta.$$

## § 1·16. Relation between Jacobi elliptic functions and their derivatives.

(a) (i) From relations (iii) of § 1·10. we have

$$1-S^2(z)=1-\frac{(e_1-e_2)}{p(z)-e_2}=\frac{p(z)-e_1}{p(z)-e_2}=C^2(z)$$

$$\therefore\quad C(z)=\{1-S^2(z)\}^{1/2}. \qquad \text{...(I)}$$

(ii) Also $\quad 1-D^2(z)=1-\dfrac{p(z)-e_3}{p(z)-e_2}$

$$=\frac{e_3-e_2}{p(z)-e_2}$$

$$=\frac{e_3-e_2}{e_1-e_2}\cdot S^2(z)$$

or $$D^2(z)=1-\frac{e_3-e_2}{e_1-e_2}S^2(z)$$

$$\therefore\ D(z)=\left\{1-\frac{e_3-e_2}{e_1-e_2}S^2(z)\right\}^{1/2}$$

Relation (2) is also written as

$$D(z)=\{1-k^2S^2(z)\}^{1/2}$$

where $$k=-\left(\frac{e_3-e_2}{e_1-e_2}\right)^{1/2}$$

$k$ is called the **modulus** of the functions $S(z)$, $C(z)$ and $D(z)$

**(iii) Also** $S'(z)=(e_1-e_2)^{1/2}\,C(z)\,D(z)$ ...(3)

$C'(z)=-(e_1-e_2)^{1/2}\,D(z)\,S(z)$ ...(4)

$D'(z)=-(e_1-e_2)^{1/2}\,k^2S(z)\,C(z)$ ...(5)

**for proof see § 1·10.**

**(b) (i)** From (v), (vi) and (vii) of § 1·10, we have

$$s_n(u)=S\{(e_1-e_2)^{-1/2}u\}$$

$$c_n(u)=C\{(e_1-e_2)^{-1/2}u\}$$

and $$d_n(u)=D\{(e_1-e_2)^{-1/2}u\}$$

$$\therefore\ c_n^2(u)=C^2\{(e_1-e_2)^{1/2}u\}^2$$

$$=1-S^2\{(e_1-e_2)^{-1/2}u\}\quad\text{from (1)}$$

$$=1-s^2_n(u).$$

$$\therefore\ c^2_n(u)=1-s^2_n(u)\}^{1/2}\quad\textbf{[Kanpur 87]}\quad ...(6)$$

(ii) Again $d_n^2(u)=D^2\{(e_1-e_2)^{-1/2}u\}$

$$=1-k^2S^2\{(e_1-e_2)^{1/2}u\}\quad\text{from (2)}$$

$$=1-k^2s^2_n(u).$$

$$\therefore\ dn(u)=\{1-k^2s_n^2(u)\}^{1/2}\quad ...(7)$$

or $$k^2s_n^2(d)\pm d_n^2(u)=1.\quad\textbf{[Kanpur 71, 87]}$$

(iii) From $s_n(u)=S\{(e_1-e_2)^{-1/2}u\}$

$$\frac{d}{dn}s_n(u)=S'\{(e_1-e_2)^{1/2}u\}.(e_1-e_2)^{-1/2}$$

$$=C\{(e_1-e_2)^{-1/2}u\}\,.\,D\{(e_1-e_2)^{-1/2}u\}\quad\text{from (3)}$$

or $$\frac{d}{du}s_n(u)=c_n(u).dn(u)\quad ...(8)$$

Similarly $$\frac{d}{du}c_n(u)=-dn(u).s_n(u)\quad ...(9)$$

and $$\frac{d}{du} dn(u) = -k^2 s_n(u)\, c_n(u). \qquad ...(10)$$

**§ 1·17. The Complementary modulus.**

The complementary modulus $k'$ associated with $S(z)$, $C(z)$ and $D(z)$ is defined by

$$k' = \left(\frac{e_1 - e_3}{e_1 - e_2}\right)^{1/2}$$

The modulus $k$ of the functions $S(z)$, $C(z)$ and $D(z)$ is given by

$$k = -\left(\frac{e_3 - e_2}{e_1 - e_2}\right)^{1/2},$$

$$k^2 + k'^2 = 1.$$

**§ 1·18. Few Important results (without proof).**

1. $p(z+\omega_1) = e_1 + \dfrac{(e_1 - e_2)(e_1 - e_3)}{p(z) - e_1}$

2. $p(z+\omega_2) = e_2 + \dfrac{(e_2 - e_1)(e_2 - e_3)}{p(z) - e_2}$

3. $p(z+\omega_3) = e_3 + \dfrac{(e_3 - e_1)(e_3 - e_2)}{p(z) - e_3}$

4. $p(Z) - p(\alpha) = -\dfrac{\sigma(z-\alpha)\,\sigma(z+\alpha)}{\sigma^2(z)\,\sigma^2(\alpha)}$

5. $\sigma_r(z) = e^{-\eta_r z} \dfrac{\sigma(z+\omega_r)}{\sigma(\omega_r)}$.

**§ 1·19. Elliptic Integrals.**

The integrals

$$Z = \int_0^{\omega} \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}} \qquad |k| < 1 \qquad ...(i)$$

is called an **ellitic integral of the first kind.** The integral exists if $\omega$ is real and such that $|w| < 1$. By analytical continuation it can be extended to other values of $\omega$.

If $t = \sin\theta$ and $\omega = \sin\phi$, the integral (i) assumes an equivalent form

$$Z = \int_0^{\phi} \frac{d\theta}{\sqrt{(1-k^2 \sin^2\theta)}} \qquad ..(ii)$$

where we often write $\qquad \phi = am\ z.$

Now if $k=0$, from (i), we have

$$\int_0^{\omega} \frac{dt}{\sqrt{(1-t^2)}} = \sin^{-1}\omega$$

$$\therefore \quad \omega = \sin Z,$$

$\therefore$ When $k \neq 0$, the integral (i) is denoted by $s_n^{-1}(\omega/k)$ or briefly $s_n^{-1}(\omega)$, when $k$ does not change during a given discussion.

Thus we have

$$Z = s_n^{-1}(\omega) = \int_0^{\omega} \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}} \quad ...(1)$$

This gives the function $\omega = s_n(z)$,

Which is called a **Jacobi's elliptic function**,

**§ 1·20. Derivatives of $s_n(z)$, $c_n(z)$ and $d_n(z)$ (another method).**

**To prove**

(a) $\frac{d}{dz}\{s_n(z)\} = c_n(z)\, d_n(z)$.

**[Kanpur 70]**

(b) $\frac{d}{dz}\{c_n(z)\} = -s_n(z).d_n(z)$.

**[Kanpur 72]**

(c) $\frac{d}{dz}\{d_n(z)\} = -k^2\, s_n(z)\, c_n(z)$.

**Proof.** (a) By definition,

if $$z = \int_0^{\omega} \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}}$$

then $$\omega = s_n(z).$$

$$\therefore \quad \frac{d}{dz}\{s_n(z)\} = \frac{d\omega}{dz} = \frac{1}{\frac{dz}{d\omega}} = \sqrt{\{(1-\omega^2)(1-k^2\omega^2)\}}$$

$$= \sqrt{[\{1-s_n^2(z)\}\{1-k^2 s_n^2(z)\}]}$$

$$= \sqrt{\{c_n^2(z).d_n^2(z)\}} \text{ from (6), (7) § 1·11.}$$

or $$\frac{\mathbf{d}}{\mathbf{dx}} \mathbf{s_n(z) = c_n(z).d_n(z)}.$$

(b) We have

$$c_n(z) = \{1 - S_n^2(z)\}^{1/2}$$

$$\therefore \quad \frac{d}{dz}\{c_n(z)\} = \frac{d}{dz}\{1 - s_n^2(z)\}^{1/2}$$

$$= \tfrac{1}{2}\{1 - s_n^2(z)\}^{-1/2}. \frac{d}{dz}\{-s_n^2(z)\}$$

$$= \tfrac{1}{2}\{c_n(z)\}^{-1}\left\{-2s_n(z).\frac{d}{dz}s_n(z)\right\}$$

$$= \tfrac{1}{2}\{c_n(z)\}^{-1}\{-2s_n(z).c_n(z).d_n(z)\} \text{ from } (a)$$

or $$\frac{\mathbf{d}}{\mathbf{dz}}\{\mathbf{c_n(z)}\} = -\mathbf{s_n(z).d_n(z)}$$

(c) we have $d_n(z) = \{1 - k^2\, s_n^2(z)\}^{1/2}$

$$\therefore \quad \frac{d}{dz} d_n(z) = \frac{d}{dz}\{1-k^2 s_n^2(z)\}^{1/2}$$

$$=\tfrac{1}{2}\{1-k^2 s_n^2(z)\}^{-1/2}\cdot\frac{d}{dz}\{-k^2 s_n^2(z)\}$$

$$=\tfrac{1}{2}\{d_n(z)\}^{-1}\{-2k^2 s_n(z).(d/dz)\, s_n(z)\}$$

$$=\tfrac{1}{2}\{d_n(z)\}^{-1}.\{-2k^2 s_n(z).c_n(z).d_n(z)\}$$

from (*a*)

$$\therefore \quad \frac{\mathbf{d}}{\mathbf{dz}}\{\mathbf{d}_n(\mathbf{z})\} = -\mathbf{k}^2\, \mathbf{s}_n(\mathbf{z}).\mathbf{c}_n(\mathbf{z}).$$

**§ 1.21. Periods of $s_n(z)$, $c_n(z)$, $d_n(z)$.**

$K$ and $iK'$ are called the quarte-periods of Jacobi's elliptic functions (See § 1·10), they are given by

$$K=\int_0^1 \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}}=\int_0^{\pi/2}\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}$$

(Putting $t=\sin\theta$)

and $$K'=\int_0^1 \frac{dt}{\sqrt{\{(1-t^2)(1-k'^2t^2)\}}}=\int_0^{\pi/2}\frac{d\theta}{\sqrt{(1-k'^2\sin^2\theta)}}$$

(Putting $t=\sin\theta$)

where $k$ and $k'$ are modulus and complementary modulus respectively s.t. $k^2+k'^2=1$.

Periods of $s_n(z)$ are $4K$ and $2iK'$

Periods of $c_n(z)$ are $K$ and $2K+2iK'$

and Periods of $d_n(z)$ are $2K$ and $4iK'$.

**§ 1·22. The Addition Theorem.**

(*a*) $$s_n(u+v)=\frac{s_n(u)\, c_n(v)\, d_n(v)+c_n(u)\, d_n(u)\, s_n(v)}{1-k^2 s_n^2(u)\, s_n^2(v)}$$

[Kanpur 8?, 85]

(*b*) $$c_n(u+v)=\frac{c_n(u)\, c_n(v)-s_n(u)\, s_n(v)\, d_n(u)\, d_n(v)}{1-k^2 s_n^2(u)\, s_n^2(v)}$$ [Kanpur 82]

[Kanpur 71]

and (*c*) $$d_n(u+v)=\frac{d_n(u)d_n(v)-k^2 s_n(u)\, s_n(v)\, c_n(u)\, c_n(v)}{1-k^2 s_n^2(u)\, s_n^2(v)}$$

**Proof.** Let $u+v=c$ (constant)

$$\therefore \quad \frac{dv}{du}=-1.$$

Let $U=s_n(u)$ and $V=s_n(v)$

$$\therefore \quad \frac{dU}{du}=\frac{d}{du}s_n(u)=c_n(u)\, d_n(u)$$

and $$\frac{dV}{du}=\frac{dV}{dv}\cdot\frac{dv}{du}=\frac{d}{dv}\{s_n(v)\}.(-1)=-c_n(v)\, d_n(v).$$

Denoting the differentials w.r.t. '$u$' by dots, we have

$$\dot{U} = c_n(u)\, d_n(u) \qquad \ldots(i)$$

and
$$\dot{V} = -c_n(v)\, d_n(v). \qquad \ldots(ii)$$

$$\therefore \quad \dot{U}^2 = c_n^2(u)\, d_n^2(u)$$
$$= \{1 - s_n^2(u)\}\{1 - k^2 s_n^2(u)\}$$

or
$$\dot{U}^2 = (1 - U^2)(1 - k^2U^2). \qquad \ldots(iii)$$

Similarly $\dot{V}^2 = (1 - V^2)(1 - k^2V^2)$. ...(iv)

Differentiating (iii) w.r.t. $u$, we have

$$2\dot{U}\ddot{U} = -2U\dot{U}(1 - k^2U^2) - 2k^2U\dot{U}(1 - U^2)$$

or
$$\ddot{U} = -(1 + k^2)\, U + 2k^2U^3. \qquad \ldots(v)$$

Similarly $\ddot{V} = -(1 + k^2)\, V + 2k^2 V^3$. ...(vi)

Multiplying (v) by $V$ and (vi) by $U$ and subtracting, we have

$$\ddot{U}V - U\ddot{V} = 2k^2\, UV\, (U^2 - V^2). \qquad \ldots(vii)$$

Again multiplying (iii) by $V^2$ and (iv) by $U^2$ and subtracting, we have

$$\dot{U}^2V^2 - U^2\dot{V}^2 = (1 - U^2)(1 - k^2U^2)V^2 - (1 - V^2)(1 - k^2V^2)U^2$$
$$= (V^2 - U^2) + k^2U^4V^2 - k^2U^2V^4$$

or
$$(\dot{U}V - U\dot{V})(\dot{U}V + U\dot{V}) = (V^2 - U^2)\{1 - k^2U^2V^2\}$$

or
$$\dot{U}V - U\dot{V} = \frac{(V^2 - U^2)(1 - k^2U^2V^2)}{\dot{U}V + U\dot{V}} \qquad \ldots(viii)$$

Dividing (vii) by (viii), we have

$$\frac{\ddot{U}V - U\ddot{V}}{\dot{U}V - U\dot{V}} = \frac{-2k^2UV(\dot{U}V + U\dot{V})}{1 - k^2U^2V^2}$$

or
$$\frac{d\,(\dot{U}V - U\dot{V})}{\dot{U}V - U\dot{V}} = \frac{d\,(1 - k^2U^2V^2)}{1 - k^2U^2V^2}$$

Integrating, we have

$$\log(\dot{U}V - U\dot{V}) = \log(1 - k^2U^2V^2) + \log A$$

$$\therefore \quad \frac{\dot{U}V - U\dot{V}}{1 - k^2U^2V^2} = A \text{ (constant)}$$

or $$\frac{c_n(u)\,d_n(u)\,s_n(v)+s_n(u)\,c_n(v)\,d_n(v)}{1-k^2\,s_n^2(u)\,s_n^2(v)}=A$$

which is the solution of differential equation (vii).

Also $u+v=c$ is a solution.

Hence

$$\frac{s_n(u)\,c_n(v)\,d_n(v)+c_n(u)\,d_n(u)\,s_n(v)}{1-k^2\,s_n^2(u)\,s_n^2(v)}=f(u+v). \quad \ldots(x)$$

Putting $v=0$, we have

$$s_n(u)=f(u)$$

$$\therefore \quad f(u+v)=s_n(u+v).$$

Hence from (ix), we have

$$s_n(u+v)=\frac{s_n(u)\,c_n(v)\,d_n(v)+c_n(u)\,d_n(u)\,s_n(v)}{1-k^2 s_n^2(u)\,s_n^2(v)}. \quad \textbf{Pd.}$$

(*b*) $c_n^2(u+v)=1-s_n^2(u+v)$

$$=1-\frac{\{s_n(u)c_n(v)\,d_n(v)+c_n(u)\,d_n(u)\,s_n(v)\}^2}{\{1-k^2\,s_n^2(u)\,s_n^2(v)\}^2}$$

$$=\frac{\{1-k^2\,s_n^2(u)\,s_n^2(v)\}^2-\{s_n(u)\,c_n(v)\,d_n(v)+c_n(u)\,d_n(u)\,s_n(v)\}^2}{\{1-k^2\,s_n^2(u)\,s_n^2(v)\}^2}$$

$$\begin{aligned}
\therefore \quad N_r &= 1+k^4\,s_n^4(u)\,s_n^4(v)-2k^2\,s_n^2(u)\,s_n^2(v)\\
&\quad -s_n^2(u)\,c_n^2(v)\,d_n^2(v)-c_n^2(u)\,d_n^2(u)\,s_n^2(v)\\
&\quad -2s_n(u)\,s_n(v)\,c_n(u)\,c_n(v)\,d_n(u)\,d_n(v)\\
&= 1+\{1-d_n^2(u)\}\{1-d_n^2(v)\}\,s_n^2(u)\,s_n^2(v)-\{1-d_n^2(u)\}\,s_n^2(v)\\
&\quad -\{1-d_n^2(v)\}\,s_n^2(u)-s_n^2(u)\,c_n^2(v)\,d_n^2(v)\\
&\quad -c_n^2(u)\,d_n^2(u)\,s_n^2(v)\\
&\quad -2s_n(u)\,s_n(v)\,c_n(u)\,c_n(v)\,d_n(u)\,d_n(v)\\
&= 1+s_n^2(u)\,s_n^2(v)+d_n^2(u)\,d_n^2(v)s_n^2(u)\,s_n^2(v)\\
&\quad -d_n^2(u)\,s_n^2(u)\,s_n^2(v)-d_n^2(v)s_n^2(u)\,s_n^2(v)-s_n^2(v)\\
&\quad +d_n^2(u)\,s_n^2(v)-s_n^2(u)+d_n^2(v)\,s_n^2(u)\\
&\quad -s_n^2(u)\,c_n^2(v)\,d_n^2(v)-c_n^2(u)\,d_n^2(u)\,s_n^2(v)\\
&\quad -2s_n(u)\,s_n(v)\,c_n(u)\,c_n(v)\,d_n(u)\,d_n(v)\\
&= \{1-s_n^2(u)-s_n^2(v)+s_n^2(u)\,s_n^2(v)\}\\
&\quad -d_n^2(u)\,s_n^2(v)\{s_n^2(u)+c_n^2(u)\}\\
&\quad -d_n^2(v)\,s_n^2(u)\{s_n^2(v)+c_n^2(v)\}\\
&\quad +s_n^2(u)\,s_n^2(v)\,d_n^2(u)\,d_n^2(v)+d_n^2(u)\,s_n^2(v)\\
&\quad +d_n^2(v)\,s_n^2(u)-2s_n(u)\,s_n(v)\,c_n(u)\,c_n(v)\,d_n(u)\,d_n(v)\\
&= \{1-s_n^2(u)\}\{1-s_n^2(v)\}+s_n^2(u)\,s_n^2(v)\,d_n^2(u)\,d_n^2(v)\\
&\quad -2s_n(u)\,s_n(v)\,c_n(u)\,c_n(v)\,d_n(u)\,d_n(v)
\end{aligned}$$

$$=\{c_n(u)\,c_n(v)-s_n(u)\,d_n(u)\,d_n(v)\}^2$$

$$\therefore \quad c_n^2(u+v)=\frac{\{c_n(u)\,c_n(v)-s_n(u)\,s_n(v)\,d_n(u)\,d_n(v)\}^2}{\{1-k^2 s_n^2(u)\,s_n^2(v)\}^2}$$

or $$c_n(u+v)=\frac{c_n(u)c_n(v)-s_n(u)s_n(v)\,d_n(u)\,d_n(v)}{1-k^2 s_n^2(u)\,s_n^2(v)}$$ **Pd.**

(c) $d_n^2(u+v)=1-k^2 s_n^2(u+v)$

$$=1-k^2\frac{\{s_n(u)\,c_n(v)\,d_n(v)+c_n(u)d_n(u)s_n(v)\}^2}{\{1-k^2s_n^2(u)\,s_n^2(v)\}^2}$$

$$=\frac{\{1-k^2s_n^2(u)s_n^2(v)\}^2-k^2\{s_n(u)\,c_n(v)\,d_n(v)+c_n(u)\,d_n(u)s_n(v)\}^2}{\{1-k^2 s_n^2(u)\,s_n^2(v)\}^2}$$

$$\therefore \quad N_r=1+k^4 s_n^4(u)\,s_n^4(v)-k^2s_n^2(u)\,s_n^2(v)$$
$$-k^2 s_n^2(u)\,c_n^2(v)\,d_n^2(v)-k^2 c_n^2(u)\,d_n^2(u)\,s_n^2(v)$$
$$-2k^2 s_n(u)\,s_n(v)\,c_n(u)\,c_n(v)\,d_n(u)\,d_n(v)$$

$$=1+k^4 s_n^2(u)\,s_n^2(v)\,\{1-c_n^2(u)\}\,\{1-c_n^2(v)\}$$
$$-k^2 s_n^2(u)\,\{1-c_n^2(u)\}-k^2 s_n^2(v)\,\{1-c_n^2(v)\}$$
$$-k^2 s_n^2(u)\,c_n^2(v)\,(1-k^2 s_n^2(v)\}-k^2c_n^2(u)\,s_n^2(v)\,\{1-k^2 s_n^2(u)\}$$
$$-2k^2 s_n(u)\,s_n(v)\,c_n(u)\,c_n(v)\,d_n(u)\,d_n(v)$$

$$=\{1+k^4 s_n^2(u)\,s_n^2(v)-k^2 s_n^2(u)-k^2 s_n^2(v)\}$$
$$+k^4 s_n^2(u)\,s_n^2(v)\,c_n^2(u)\,c_n^2(v)$$
$$-2k^2 s_n(u)\,s_n(v)\,c_n(u)\,c_n(v)\,d_n(u)d_n(v)$$

$$=\{1-k^2 s_n^2(u)\}\,\{1-k^2 s_n^2(v)\}+k^4 s_n^2(u)\,s_n^2(v)\,c_n^2(u)\,c_n^2(v)$$
$$-2k^2 s_n(u)\,s_n(v)\,c_n(u)\,c_n(v)\,d_n(u)\,d_n(v)$$

$$=d_n^2(u)\,d_n^2(v)+k^4 s_n^2(u)\,s_n^2(v)\,c_n^2(u)\,c_n^2(v)$$
$$-2k^2 s_n(u)\,s_n(v)\,c_n(u)\,c_n(v)\,d_n(u)\,d_n(v)$$

$$=\{d_n(u)\,d_n(v)-k^2 s_n(u)\,s_n(v)\,c_n(u)\,c_n(v)\}^2$$

$$\therefore \quad d_n^2(u+v)=\frac{\{d_n(u)\,d_n(v)-k^2 s_n(u)\,s_n(v)\,c_n(u)\,c_n(v)\}^2}{\{1-k^2 s_n^2(u)\,s_n^2(v)\}^2}$$

or $$d_n(u+v)=\frac{d_n(u)\,d_n(v)-k^2 s_n(u)\,s_n(v)\,c_n(u)\,c_n(v)}{1-k^2 s_n^2(u)\,s_n^2(v)}$$ **Pd.**

### § 1·23. Value of $s_n$ (2u), $c_n$ (2u), $d_n$ (2u)

Prove that

(a) $$s_n(2u)=\frac{2s_n(u)\,c_n(u)\,d_n(u)}{1-k^2 s_n^4(u)}$$

(b) $$c_n(2u)=\frac{c_n(u)-s_n^2(u)\,c_n^2(u)}{1-k^2 s_n^4(u)}$$

and (c) $$d_n(2u)=\frac{d_n^2(u)-k^2 s_n^2(u)\,d_n^2(u).}{1-k^2 s_n^4(u)}$$ [Kanpur 71]

**Proof.** Replace $v$ by $u$ in the result of § 1·17.

## ILLUSTRATIVE EXAMPLES

**Ex. 1.** *Show that*

$$\eta_1\omega_2-\eta_2\omega_1=\eta_2\omega_3-\eta_3\omega_2=\eta_3\omega_1-\eta_1\omega_3=\frac{\pi i}{2}$$

*with their usual notations.*

**Sol.** Since the only singularities of $\zeta(z)$ within or on the cell $C$ of vertices $\omega_1-\omega_2,\ \omega_1+\omega_2\ -\omega_1+\omega_2,\ -\omega_1-\omega_2$ is simple pole at the origin of residue 1, hence we have

$$(2\pi i).1=\int_C \zeta(z)\,dz$$

$$=\int_{\omega_1-\omega_2}^{\omega_1+\omega_2}\zeta(z)\,dz+\int_{\omega_1+\omega_2}^{-\omega_1+\omega_2}\zeta(z)\,dz$$

$$+\int_{-\omega_1+\omega_2}^{-\omega_1-\omega_2}\zeta(z)\,dz+\int_{-\omega_1-\omega_2}^{\omega_1-\omega_2}\zeta(z)dz$$

$$=\int_{\omega_1-\omega_2}^{\omega_1+\omega_2}\zeta(z)\,dz+\int_{\omega_1-\omega_2}^{-\omega_1-\omega_2}\zeta(t+2\omega_2)\,dt$$

$$+\int_{\omega_1+\omega_2}^{\omega_1-\omega_2}\zeta(u-2\omega_1)\,du+\int_{-\omega_1-\omega_2}^{\omega_1-\omega_2}\zeta(z)\,dz$$

Putting $z=t+2\omega_2$ in second integral and $z=u-2\omega_1$ in third integral.

$$=\int_{\omega_1-\omega_2}^{\omega_1+\omega_2}\{\zeta(z)-\zeta\ z-(2\omega_1)\}\,dz$$

$$+\int_{\omega_1-\omega_2}^{-\omega_1-\omega_2}\{\zeta(z+2\omega_2)-\zeta(z)\}\,dz$$

$$=2\eta_1\int_{\omega_1-\omega_2}^{\omega_1+\omega_2}dz+2\eta_2\int_{\omega_1-\omega_2}^{-\omega_1-\omega_2}dz=4\eta_1\omega_2-4\eta_2\omega_1.$$

$$\therefore\quad \eta_1\omega_2-\eta_2\omega_1=\frac{\pi i}{2}.$$

Similarly prove other results.

**Ex. 2.** *Show that* $s_n(0)=0,\ c_n(0)=0, d_n(0)=0.$

**Sol.** We have

$$s_n(kz/\omega_1)=\frac{k\sigma(z)}{\omega_1\,\sigma_2(z)};\ c_n\left(\frac{kz}{\omega_1}\right)=\frac{\sigma_1(z)}{\sigma_2(z)}$$

and $$d_n\left(\frac{kz}{\omega_1}\right)=\frac{\sigma_3(z)}{\sigma_2(z)}$$

$$\sigma(z)=z\prod_{m,n}\left\{1-\frac{z}{\Omega_{m,n}}\right\}.\exp.\left(\frac{z}{\Omega_{m,n}}+\frac{z^2}{2\Omega^2_{m,n}}\right)\}$$

$$\sigma_r(z)=e^{\frac{-\eta_r z\,\sigma(z+\omega_r)}{\sigma(\omega_r)}} \qquad (r=1, 2, 3)$$

and

when $z=0$ ; $\sigma(z)=0$, $\sigma_r(z)=1$.

$\therefore$ $s_n(0)=0$, $c_n(0)=1$, $d_n(0)=1$. **Pd.**

Ex. 3. *Prove that (a) $s_n(-z)=-s_n(z)$, (b) $c_n(-z)=c_n(z)$ and (c) $d_n(-z)=d_n(z)$.*

**Sol.** (a) We have

If $$z=\int_0^{\omega}\frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}}$$

then $$\omega=s_n(z).$$

Putting $t=-u$ so that $dt=-du$, we have

$$z=-\int_0^{-\omega}\frac{du}{\sqrt{\{(1-u^2)(1-k^2u^2)\}}}$$

or $$-z=\int_0^{-\omega}\frac{du}{\sqrt{\{(1-u^2)(1-k^2u^2)\}}}$$

$\therefore$ $s_n(-z)=-\omega$

or $s_n(-z)=-s_n(z)$. **Pd.**

(b) $$c_n(-z)=\sqrt{\{1-s_n^2(-z)\}}=\sqrt{[1-\{-s_n(z)\}^2]}$$
$$=\sqrt{[1-\{s_n(z)\}^2]}=c_n(z)$$

and (c) $$d_n(-z)=\sqrt{\{1-k^2s_n^2(-z)\}}=\sqrt{[1-k^2\{-s_n(z)\}^2]}$$
$$=\sqrt{\{1-k^2s_n^2(z)\}}=d_n(z).$$ **Pd.**

**Ex.** *Prove that*

(a) $s_n(K+iK')=\dfrac{1}{k}$

(b) $c_n(K+iK')=-\dfrac{ik_r}{k}$,

and (c) $d_n(K+iK')=0$.

**Sol.** (a) We have

$$K=\int_0^1\frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}}$$

and $$K'=\int_0^1\frac{dt}{\sqrt{\{(1-t^2)(1-k'^2t^2)\}}}$$ where $k'^2+k^2=1$.

Putting $\dfrac{1}{\sqrt{(1-k'^2t^2)}}=u$

so that $t^2=\dfrac{u^2-1}{k'^2u^2}$

and $$dt=\frac{du}{k'u^2\sqrt{(u^2-1)}}$$

$$K' = \int_0^{1/k} \frac{du}{\sqrt{\{(u^2-1)(1-k^2u^2)\}}}$$

or $$K' = -i \int_0^{1/k} \frac{du}{\sqrt{\{(1-u^2)(1-k^2u^2)\}}}$$

$$\therefore\ K+iK' = \int_0^1 \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}} + \int_1^{1/k} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}$$

$$= \int_0^1 \frac{du}{\sqrt{\{(1-u^2)(1-k^2u^2)\}}} + \int_1^{1/k} \frac{du}{\sqrt{\{(1-u^2)(1-k^2u^2)\}}}$$

$$= \int_0^{1/k} \frac{du}{\sqrt{\{(1-u^2)(1-k^2u^2)\}}}$$

$\therefore\ s_n(K+iK') = \dfrac{1}{k}$. Since if $z = \displaystyle\int_0^{\omega} \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}}$

then $\omega = s_n(z)$.

(*b*) $c_n(K+iK') = \sqrt{\{1-s_n^2(K+iK')\}} = \sqrt{\left(1-\dfrac{1}{k^2}\right)}$

$$= \frac{\sqrt{(k^2-1)}}{k} = \frac{\sqrt{(-k'^2)}}{k}$$

$$= \frac{ik'}{k}.$$

(*c*) $d_n(K+iK') = \sqrt{\{1-k^2 s_n^2(K+iK')\}} = \sqrt{\left(1-k^2 \cdot \dfrac{1}{k^2}\right)}$ from (*a*)

$= 0$. **Pb.**

**Ex. 5.** *Prove that*

(*a*) $s_n(z+2k) = -s_n(z)$,

(*b*) $c_n(z+2k) = -c_n(z)$

*and* (*c*) $d_n(z+2K) = d_n(z)$

*and hence deduce that*

(*a*) $s_n(z+4K) = s_n(z)$

*and* (*b*) $c_n(z+4K) = c_n(z)$.

**Sol.** We have

$$z = \int_0^{\phi} \frac{d\theta}{\sqrt{(1-k^2 \sin^2 \theta)}},\ \text{so that } \phi = a_m z$$

and $\sin \phi = s_n(z)$, $\cos \phi = c_n(z)$.

Now $$\int_0^{\phi+\pi} \frac{d\theta}{\sqrt{(1-k^2 \sin^2 \theta)}} = \int_0^{\pi} \frac{d\theta}{\sqrt{(1-k^2 \sin^2 \theta)}}$$

$$+\int_{\pi}^{\phi+\pi}\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}$$

$$=2\int_0^{\pi/2}\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}+\int_0^{\phi}\frac{d\theta}{\sqrt{(1-k^2\sin^2 u)}}$$

$$=2K+z$$

Putting $\theta=\pi+u$
so that $d\theta=du$

$\therefore\ \phi+\pi=a_m(z+2K)$

Hence

(a) $s_n(z+2k)=\sin\{a_m(z+2k)\}=\sin(\phi+\pi)$

$\quad = -\sin\phi=-s_n(z)$

(b) $c_n(z+2K)=\cos\{a_m(z+k)\}=\cos(\phi+\pi)$

$\quad = -\cos\phi=-c_n(z)$

and (c) $d_n(z+2K)=\sqrt{\{1-k^2 s_n^2(z+2K)\}}$

$\quad =\{1-k^2 s_n^2(z)\}=d_n(z).$

**Deductions.**

(a) $s_n(z+4K)=s_n(z+2K+2K)=-s_n(z+2K)$

$\quad =-\{-s_n(z)\}$

$\quad =s_n(z)$

and (b) $c_n(z+4K)=c_n(z+2K+2K)$

$\quad =-c_n(z+2K)$

$\quad =-\{-c_n(z)\}$

$\quad =c_n(z).$ **Pd.**

## EXERCISE ON CHAPTER I

1. Prove that

$$p(2z)=\frac{1}{4}\left\{\frac{p''(z)}{p'(z)}\right\}^2-2p(z).$$

2. Prove that

$$\sigma(2z)=-\frac{2\{\sigma(z)\,\sigma(z-\omega_1)\,\sigma(z-\omega_2)\,\sigma(z-\omega_3)\}}{\sigma(\omega_1)\,\sigma(\omega_2)\,\sigma(\omega_3)}$$

3. Show that, if $3\alpha$ be a period of $p(z)$

$\{p(z)-p(\alpha)\}\{p(z+\alpha)-p(\alpha)\}\{p(z+2\alpha)\}-p(\alpha)\}=\{-p'^2(\alpha)\}.$

4. Prove that

$$(a)\ s_n^2(u)=\frac{1-c_n(2u)}{1+d_n(2u)},\quad (b)\ c_n^2(u)=\frac{d_n(2u)+c_n(2u)}{1+d_n(2u)}$$

$$\text{and } (c)\ d_n^2(u)=\frac{d_n(2u)+c_n(2u)}{1+c_n(2u)}$$

5. Show that

(i) $s_n(u+v)\,c_n(u-v)+s_n(u-v)\,c_n(u+v)$

$$\frac{2s_n(u)\,c_n(u)\,d_n(v)}{1-k^2s_n^2(u)\,s_n^2(v)}$$

(ii) $c_n(u+v)\,c_n(u-v)\,s_n(u+v)\,s_n(u-v)$

$$=\frac{c_n^2(v)-s_n^2(v)\,d_n^2(u)}{1-k^2\,s_n^2(u)\,s_n^2(v)}.$$

**6.** Prove that

(*i*) $s_n(u+\alpha)\,s_n(u-\alpha)=\left\{\dfrac{s_n^2(u)-s_n^2(\alpha)}{1-k^2\,s_n^2(u)\,s_n^2(\alpha)}\right\}$.

**[Kanpur 85]**

(*ii*) $c_n(u+\alpha)\,c_n(u-\alpha)=\left\{\dfrac{c_n^2(\alpha)-d_n^2(\alpha)\,s_n^2(u)}{1-k^2\,s_n^2(u)\,s_n^2(\alpha)}\right\}$

and (*iii*) $d_n(u+\alpha)\,d_n(u-\alpha)=\left\{\dfrac{d_n^2(\alpha)-k^2\,c_n^2(\alpha)\,s_n^2(u)}{1-k^2\,s_n^2(u)\,s_n^2(\alpha)}\right\}$

**7.** Prove that

(i) $s_n(z+2iK')=s_n(z)$, (ii) $c_n(z+2K+2iK')=c_n(z)$

and (iii) $d_n(z+4iK')=d_n(z)$.

2

# Beta and Gamma Functions

## § 2·1. Principal and general values of an improper integral.

[Agra 63]

Consider the integral $\int_a^b f(x)\, dx$

in which $f(x)$, becomes infinite at $x=c$, where $c$ lies between the limits $a$ and $b$. In this case we define the integral,

$$\int_a^b f(x)\, dx$$

by
$$\lim_{\epsilon\to 0}\left[\int_a^{c-\mu\epsilon} f(x)\, dx+\int_{c+\nu\epsilon}^b f(x)\, dx\right.$$

$\mu$ and $\nu$ being arbitrary constants.

The above limit gives what is called the **general value** of the improper integral. If $\mu=\nu$, the value of the above limits is called the **principal value** of the integral.

**Example.** *Find the general and principal values of the following integrals.*

(i) $\int_{-\pi/4}^{\pi/2} \cot x\, dx$, (ii) $\int_0^2 \frac{dx}{(x-1)^3}$

**Sol.** (i) Since $\cot x$ becomes infinite at $x=0$, we have

$$\int_{-\pi/4}^{\pi/2} \cot x\, dx=\lim_{\epsilon\to 0}\left[\int_{-\pi/4}^{-\mu\epsilon} \cot x\, dx+\int_{\nu\epsilon}^{\pi/2} \cot x\, dx\right]$$

$$=\lim_{\epsilon\to 0}\left[\log \sin(-\mu\epsilon)-\log \sin\left(-\frac{\pi}{4}\right)\right.$$

$$\left.+\log \sin\left(\frac{\pi}{2}\right)-\log \sin \nu\epsilon\right]$$

$$=\lim_{\epsilon\to 0}\left[\log\left\{\frac{\sin(-\mu\epsilon)}{\sin(-\pi/4)}\times\frac{1}{\sin \nu\epsilon}\right\}\right]$$

$$=\lim_{\epsilon\to 0}\log\left[\frac{\sqrt{2}\sin \mu\epsilon}{\sin \nu\epsilon}\right]$$

$$=\lim_{\epsilon\to 0}\log\left[\sqrt{2}\,\frac{\sin\mu\epsilon}{\mu\epsilon}\times\frac{\nu\epsilon}{\sin\nu\epsilon}\times\frac{\mu}{\nu}\right]$$

$$=\log\frac{\sqrt{2}\mu}{\nu}$$

which is the general value of the integral.

The principal value $=\log\sqrt{2}=\frac{1}{2}\log 2.$

(ii) Here $\dfrac{1}{(x-1)^3}$ becomes infinite at $x=1$. Therefore we have

$$\int_0^2\frac{1}{(x-1)^3}\,dx=\lim_{\epsilon\to 0}\left[\int_0^{1-\mu\epsilon}\frac{dx}{(x-1)^3}+\int_{1+\nu\epsilon}^{2}\frac{dx}{(x-1)^3}\right]$$

$$=\lim_{\epsilon\to 0}\left[-\frac{1}{2\mu^2\epsilon^2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{2\nu^2\epsilon^2}\right]$$

$$=\lim_{\epsilon\to 0}\frac{\mu^2-\nu^2}{2\mu^2\nu^2\epsilon^2}=\infty \quad \text{if } \mu\neq\nu.$$

and $\qquad =0 \qquad$ if $\mu=\nu$.

*i.e.* the general value of the integral is finite while its principal value is zero.

**§ 2·2. Infinite limits.** If $f(x)$ is finite for all real values of $x$, then we define the integral $\int_a^{\infty} f(x)\,dx$ by $\lim_{\epsilon\to 0}\int_a^{1/\epsilon} f(x)\,dx.$

Similarly the integral $\int_{-\infty}^{b} f(x)\,dx$ is given by

$$\lim_{\epsilon\to 0}\int_{-1/\epsilon}^{b} f(x)\,dx.$$

Also $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx=\lim_{\epsilon\to 0}\int_{-1/\mu\epsilon}^{1/\nu\epsilon} f(x)\,dx$

which gives the **general value** of the integral. The principal value is obtained by putting $\mu=\nu$.

**§ 2·3. To find the value of $\displaystyle\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\,dx$, where $\dfrac{f(x)}{F(x)}$ is a rational algebraic function in which f(x) is at least two degree lower in x than $F(x)$ and all the roots of F(x)=0 are imaginary.**

Since all the roots of $F(x)=0$ are imaginary, it follows that $\dfrac{f(x)}{F(x)}$ is finite for all real values of $x$. Hence we have

$$\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\,dx=\lim_{\epsilon\to 0}\int_{-1/\mu\epsilon}^{1/\nu\epsilon}\frac{f(x)}{F(x)}\,dx$$

If $\alpha \pm \beta i$ be pair of imaginary roots of $F(x)=0$, then the corresponding partial fractions of $\dfrac{f(x)}{F(x)}$ are of the form

$$\frac{L-Mi}{x-\alpha-\beta i} \text{ and } \frac{L+Mi}{x-\alpha+\beta i}$$

Now $\dfrac{L-Mi}{x-\alpha-\beta i}+\dfrac{L+Mi}{x-\alpha+\beta i}=\dfrac{2L(x-\alpha)+2M\beta}{(x-\alpha)^2+\beta^2}$.

Again $\displaystyle\int_{-\infty}^{\infty} \frac{2M\beta}{(x-\alpha)^2+\beta^2}\,dx$

$$=\lim_{\epsilon\to 0} 2M\left[\tan^{-1}\frac{x-\alpha}{\beta}\right]_{-1/\mu\epsilon}^{1/\nu\epsilon}=2M\pi$$

and $\displaystyle\int_{-\infty}^{\infty} \frac{2L(x-\alpha)}{(x-\alpha)^2+\beta^2}\,dx=\lim_{\epsilon\to 0}\int_{-1/\mu\epsilon}^{1/\nu\epsilon} \frac{2L(x-\alpha)}{(x-\alpha)^2+\beta^2}\,dx$

$$=\lim_{\epsilon\to 0} L\left[\log\{(x-\alpha)^2+\beta^2\}\right]_{-1/\mu\epsilon}^{1/\nu\epsilon}$$

$$=\lim_{\epsilon\to 0} L\log\left\{\frac{\mu^2(1-\epsilon\alpha\nu)^2+\beta^2\epsilon^2\nu^2}{\nu^2(1+\alpha\mu\epsilon)^2+\beta^2\mu^2\epsilon^2}\right\}$$

$$=L\log\frac{\mu^2}{\nu^2}=2L\log\frac{\mu}{\nu}$$

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{2L(x-\alpha)+2M\beta}{(x-\alpha)^2+\beta^2}\,dx=2L\log\frac{\mu}{\nu}+2\pi M.$

Now, let $F(x)$ be of degree $2n$ in $x$ and the values of $L$ and $M$, corresponding to the $n$ pairs of imaginary roots, be denoted by $L_1, L_2, \ldots L_n$ and $M_1, M_2, \ldots M_n$ respectively; then we have

$$\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\,dx=2(L_1+L_2+\ldots+L_n)\log\frac{\mu}{\nu}$$
$$+2\pi(M_1+M_2+\ldots+M_n).$$

$$\frac{f(x)}{F(x)}=\frac{2L_1(x-\alpha_1)+2M_1\beta_1}{(x-\alpha_1)+\beta_1^2}+\ldots+\frac{2L_n(x-\alpha_n)+2M_n\beta_n}{(x-\alpha_n)^2+\beta_n^2}.$$

Multiplying both sides by $F(x)$, we see that the coefficient of $x^{2n-1}$ on the right hand side is clearly

$$2(L_1+L_2+\ldots+L_n)$$

And since $f(x)$ is at most of degree $2n-2$ the coefficient of $x^{2n-1}$ on the left hand side is zero.

Hence $\qquad 2(L_1+L_2+\ldots+L_n)=0.$

It follows that in this case,

$$\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\,dx=2\pi(M_1+M_2+\ldots+M_n)$$

**§ 2.4. To find the value of the integral**

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx$$

**where m and n are positive integers n > m.**

Let $a+bi=\alpha$, be a root of the equation $x^{2n}+1=0$. Then the corresponding partial fraction is of the form

$$\frac{L-Mi}{x-\alpha}$$

Now $L-Mi=\dfrac{\alpha^{2m}}{2n\alpha^{2n-1}}=\dfrac{\alpha^{2m}\alpha}{2n\,\alpha^{2n}}=\dfrac{\alpha^{2m+1}}{2n\,(-1)}$.

[**Rule**—*If* $\alpha$ *be a root of* $F(x)=0$, *then the corresponding partial fraction of* $\dfrac{f(x)}{F(x)}$ *is given by* $\dfrac{f(\alpha)}{F'(\alpha)\,(x-\alpha)}$]

Solving the equation $x^{2n}+1=0$, we have

$x=(-1)^{1/2n}=[\cos(2k+1)\pi+i\sin(2k+1)\pi]^{1/2n}$

$$=\cos\left(\frac{2k+1}{2n}\pi\right)+i\sin\left(\frac{2k+1}{2n}\pi\right)$$

where $\quad k=0, 1, 2, \ldots (2n-1)$;

or $\quad x=\cos\left(\dfrac{2k+1}{2n}\pi\right)\pm i\sin\left(\dfrac{2k+1}{2n}\pi\right)$

where $\quad k=0, 1, 2, \ldots (n-1)$.

Therefore $\alpha$ is of the form

$$\cos\left(\frac{2k+1}{2n}\pi\right)+i\sin\left(\frac{2k+1}{2n}\pi\right)$$

Hence $\alpha^{2m+1}=\cos(2k+1)\,\theta+i\sin(2k+1)\,\theta$

where $\quad \theta=\dfrac{(2m+1)\,\pi}{2n}$

It follows that $M=\dfrac{\sin(2k+1)\,\theta}{2n}$

and accordingly, we get

$$M_1+M_2+\ldots+M_n$$

$$=\frac{1}{2n}[\sin\theta+\sin 3\theta+\ldots+\sin(2n-1)\,\theta]$$

To find the sum, let

$$S=\sin\theta+\sin 3\theta+\ldots+\sin(2n-1)\,\theta.$$

Then $2S\sin\theta=2\sin^2\theta+2\sin\theta\sin 3\theta+\ldots$

$$+2\sin\theta\sin(2n-1)\,\theta$$

$$=(1-\cos 2\theta)+(\cos 2\theta-\cos 4\theta)+\ldots+\{\cos(2n-2)\theta-\cos 2n\theta\}$$

$$=1-\cos 2n\theta=2\sin^2 n\theta$$

$$=2\sin^2\left\{(2m+1)\frac{\pi}{2}\right\}=2$$

$$\therefore\quad S=\frac{1}{\sin\theta}=\frac{1}{\sin\left(\frac{2m+1}{2n}\pi\right)}$$

Consequently, we have

$$\int_{-\infty}^{\infty}\frac{x^{2m}}{1+x^{2n}}dx=2\pi\,(M_1+M_2+\ldots+M_n)$$

$$=\frac{\pi}{n\sin\frac{2m+1}{2n}\pi}.$$

Hence $\displaystyle\int_0^{\infty}\frac{x^{2m}}{1+x^{2n}}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^{2m}}{1+x^{2n}}dx=\frac{\pi}{2n\sin\frac{2m+1}{2n}\pi}$

**§ 2·5. To find the value of** $\displaystyle\int_0^{\infty}\frac{x^{2m}}{1-x^{2n}}dx$

Here the equation $1-x^{2n}=0$ has only two real roots *viz.* $-1$ and $1$, and the rest of the roots are imaginary and the corresponding partial fiactions are easily seen to be

$$\frac{1}{2n(x+1)}-\frac{1}{2n(x-1)}=\frac{1}{n(1-x^2)}.$$

Now we shall show that $\displaystyle\int_0^{\infty}\frac{dx}{1-x^2}=0.$

$$\int_0^{\infty}\frac{dx}{1-x^2}=\int_0^1\frac{dx}{1-x^2}+\int_1^{\infty}\frac{dx}{1-x^2}.$$

On putting $x=\dfrac{1}{z}$ in the second integral, we have

$$\int_1^{\infty}\frac{dx}{1-x^2}=\int_1^0\frac{dz}{1-z^2}=-\int_0^1\frac{dz}{1-z^2}=-\int_0^1\frac{dx}{1-x^2}$$

$$\therefore\quad\int_0^{\infty}\frac{dx}{1-x^2}=0.$$

Consequently, this part of definite integral which corresponds to the real roots disappears.

Moreover by § 2·3, we have

$$\int_{-\infty}^{\infty} \frac{x^{2m}}{1-x^{2n}}dx=2\pi\,(M_1+M_2+\ldots+M_{n-1}).$$

As in § 2·4, the roots of $x^{2n}-1=0$ are easily seen to be of the form $\cos\frac{\pi k}{n}\pm i\sin\frac{\pi k}{n}$, where $k=0, 1, 2, \ldots, n-1$.

And proceeding as in § 2·4, we have

$$M_1+M_2+\ldots+M_{n-1}$$
$$=\frac{1}{2n}\,[\sin 2\theta+\sin 4\theta+\ldots+\sin 2\,(n-1)\,\theta],$$

where $$\theta=\frac{(2m+1)\,\pi}{2n}$$

Also $\sin 2\theta+\sin 4\theta+\ldots+\sin 2\,(n-1)\,\theta$

$$=\frac{\cos\theta-\cos\,(2n-1)\,\theta}{2\sin\theta}$$

$$=\cot\frac{2m+1}{2n}\pi.$$

$$\therefore \quad \int_0^\infty \frac{x^{2m}}{1-x^{2n}}\,dx=\frac{\pi}{2n}\cot\frac{2m+1}{2n}\pi.$$

§ 2·6. **Deduction from** $\int_0^\infty \frac{x^{2m}}{1+x^{2n}}dx$.

and $$\int_0^\infty \frac{x^{2m}}{1-x^{2n}}\,dx.$$

If we put $x^{2n}=z$ and $\frac{2m+1}{2n}=a$. in the results of § 2·4 & § 2·5 we have

$$\int_0^\infty \frac{z^{a-1}}{1+z}\,dz=\frac{\pi}{\sin a\pi}\;;\;\int_0^\infty \frac{z^{a-1}}{1-z}\,dz=\pi\cot a\pi \qquad \ldots(1)$$

In this case $a$ should be less than unity.

By putting $z^a=u$ in (1), we get

$$\int_0^\infty \frac{du}{1+u^{1/a}}=\frac{a\pi}{\sin a\pi},\quad \int_0^\infty \frac{du}{1-u^{1/a}}=a\pi\cot a\pi.$$

If $\frac{1}{a}=r$, these become

$$\int_0^\infty \frac{du}{1+u^r}=\frac{\pi}{r\sin\frac{\pi}{r}}\;;\;\int_0^\infty \frac{du}{1+u^r}=\frac{\pi}{r}\cot\frac{\pi}{r} \qquad \ldots(2)$$

where $r$ is positive and greater than unity.

Again $$\int_0^\infty \frac{x^n}{1+x^2}\,dx=\int_0^1 \frac{x^n\,dx}{1+x^2}+\int_1^\infty \frac{x^n\,dx}{1+x^2}$$

Now by putting $x=\frac{1}{z}$ in $\int_0^\infty \frac{x^n}{1+x^2}\,dx$, we get

$$\int_0^\infty \frac{x^n}{1+x^2}\,dx=-\int_1^0 \frac{z^{-n}}{1+z^2}\,dz=\int_0^1 \frac{x^{-n}}{1+x^2}\,dx.$$

$$\therefore \int_0^\infty \frac{x^n\,dx}{1+x^2}=\int_0^1 \frac{x^n+x^{-n}}{1+x^2}\,dx. \qquad ...(3)$$

Also from § 2·4, when $n>1$, we get

$$\int_0^\infty \frac{x^n}{1+x^2}\,dx=\frac{\pi}{2\cos\frac{nx}{2}} \qquad ...(4)$$

Hence $$\int_0^1 \frac{x^n+x^{-n}}{x+x^{-1}}\,dx=\frac{\pi}{2\cos\frac{nx}{2}} \qquad ...(5)$$

Simliarly $$\int_0^1 \frac{x^n-x^{-n}}{x-x^{-1}}\frac{dx}{x}=\frac{\pi}{2}\tan\frac{n\pi}{2}. \qquad ...(6)$$

Again by putting $x=e^{-\pi z}$ and $n\pi=a$ in (5) and (6) we get

$$\int_0^\infty \frac{e^{az}+e^{-az}}{e^{\pi z}+e^{-\pi z}}\,dz=\tfrac{1}{2}\sec\frac{a}{2}. \qquad ...(7)$$

and $$\int_0^\infty \frac{e^{az}-e^{-az}}{e^{\pi z}-e^{-\pi z}}\,dz=\tfrac{1}{2}\tan\frac{a}{2} \qquad ...(8)$$

## EXAMPLES

**Ex. 1.** *Prove that* $\int_0^1 \frac{z^a-z^{-a}}{1-z}\,dz=\pi\cot a\pi-\frac{1}{a}$.

**Sol.** we have $\int_0^1 \frac{z^a-z^{-a}}{1-z}\,dz$

$$=\int_0^1 \frac{z^a}{1-z}\,dz-\int_0^1 \frac{z^{-a}}{1-z}\,dz$$

$$=-\int_0^1 z^{a-1}\,dz+\int_0^1 \frac{z^{-a}}{1-z}\,dz-\int_0^1 \frac{z^{-a}}{1-z}\,dz$$

Now if we put $z=\frac{1}{x}$ in $\int_0^1 \frac{z^{-a}}{1-z}\,dz$, we have

$$\int_0^1 \frac{z^{-a}}{1-z}\,dz=\int_\infty^1 \frac{x^a}{1-\frac{1}{x}}\left(-\frac{1}{x^2}\right)dx$$

$$=-\int_1^\infty \frac{x^{a-1}}{1-x}\,dx=-\int_1^\infty \frac{z^{a-1}}{1-z}\,dz.$$

$\therefore$ the given integral

$$=-\int_0^1 z^{a-1}\,dz+\int_0^1 \frac{z^{a-1}}{1-z}\,dz+\int_1^\infty \frac{z^{a-1}}{1-z}\,dz.$$

$$=-\left[\frac{z^a}{a}\right]_0^1+\int_0^\infty \frac{z^{a-1}}{1-z}\,dz$$

$$=-\frac{1}{a}+\pi\cot a\pi=\pi\cot a\pi-\frac{1}{a}.$$

**§ 2·7. Method of differentiation under the Integration Sign.** By this method, the value of a definite integral can be determined by differentiating the integrand with respect to a quantity of which the limits of integration are independent. This method can also be applied to indefinite integrals. It is a very important method of integration. This method is applied in two ways. Firstly, new integrals, can be deduced from certain known standrad integrals. Secondly, the value of given integral can be found by first differentiating the integrand, then evaluating the new integral thus obtained and finally integrating the result with respect to the same quantity with respect to which the integrand was first differentiated.

The following example, illustrates the method.

**Ex. 2.** *Evaluate the integrals.*

(*i*) $\displaystyle\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx$

(*ii*) $\displaystyle\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}\,dx.$

(*iii*) $\displaystyle\int_0^\infty \frac{\log(1+a^2x^2)}{1+a^2x^2}\,dx.$

**Sol.** (i) Let $u=\displaystyle\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx.$

Differentiating w.r.t. $a$, we have

$$\frac{du}{da}=\int_0^\infty \frac{1}{(1+x^2)(1+a^2x^2)}\,dx$$

$$=\int_0^\infty\left[\frac{1}{(1-a^2)(1+x^2)}-\frac{a^2}{(1-a^2)(1+a^2x^2)}\right]dx$$

$$=\frac{1}{1-a^2}\left[\tan^{-1}x-\frac{a}{1-a^2}\tan^{-1}ax\right]_0^\infty$$

$$=\frac{\pi}{2}\left[\frac{1}{1-a^2}-\frac{a}{1-a^2}\right]=\frac{\pi}{2(1+a)}$$

$$\therefore\quad u=\frac{\pi}{2}\int\frac{da}{1+a}=\frac{\pi}{2}\log(1+a)+A.$$

when $a=0$, $u=0$. $\therefore$ $A=0$.

Hence $\displaystyle\int_0^{\infty} \frac{\tan^{-1} ax}{x(1+x^2)}\,dx=\frac{\pi}{2}\log(1+a)$.

(ii) Let $\displaystyle u=\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}\,dx$

Then $\displaystyle\frac{du}{d\alpha}=-\int_0^{\pi/2} \frac{\sin\alpha}{1+\cos\alpha\cos x}\,dx$

$$=-\int_0^{\pi/2} \frac{\sin\alpha\,dx}{1+\cos\alpha\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$=-\sin\alpha\int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\,dx}{\left(1+\tan^2\frac{x}{2}+\cos\alpha-\cos x\tan^2\frac{x}{2}\right)}$$

$$=-\sin\alpha\int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\,dx}{2\cos^2\frac{\alpha}{2}+2\sin^2\frac{\alpha}{2}\tan^2\frac{x}{2}}$$

$$=-\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\sin^2\frac{\alpha}{2}}\int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\,dx}{\cot^2\frac{\alpha}{2}+\tan^2\frac{x}{2}}$$

$$=-\cot\frac{\alpha}{2}\cdot\frac{2}{\cot\frac{\alpha}{2}}\left(\tan^{-1}\frac{\tan\frac{x}{2}}{\cot\frac{\alpha}{2}}\right)_0^{\pi/2}$$

$$=-2.\frac{\alpha}{2}=-\alpha$$

$\therefore$ $u=\frac{1}{2}\alpha^2+A$.

When $\alpha=\frac{\pi}{2}$, $u=0$. $\therefore$ $A=\frac{\pi^2}{8}$.

Hence $\displaystyle\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos\alpha}\,dx=\frac{1}{2}\left(\frac{\pi^2}{4}-\alpha^2\right)$

(iii) Let $\displaystyle u=\int_0^{\infty} \frac{\log(1+a^2x^2)}{1+b^2x^2}\,dx$.

Then $\displaystyle\frac{du}{da}=\int_0^{\infty} \frac{2ax^2}{(1+b^2x^2)(1+a^2x^2)}\,dx$

$$=2a\int_0^{\infty}\left[\frac{1}{(b^2-a^2)}\cdot\frac{1}{(1+a^2x^2)}-\frac{1}{(b^2-a^2)}\cdot\frac{1}{(1+b^2x^2)}\right]dx$$

$$=\frac{2a}{b^2-a^2}\left[\frac{1}{a}\tan^{-1}ax-\frac{1}{b}\tan^{-1}bx\right]_0^\infty$$

$$=\frac{2a}{b^2-a^2}\left(\frac{1}{a}-\frac{1}{b}\right)\frac{\pi}{2}=\frac{\pi}{b\,(a+b)}.$$

Therefore $u=\frac{\pi}{b}\log(a+b)+A$.

Now when $a=0$, $u=0$.

$\therefore\ A=-\frac{\pi}{b}\log b$.

Hence $\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}\,dx=\frac{\pi}{b}\log\frac{a+b}{b}$.

**§ 2·8. Method of Integration under the sign of integration.**

This method is similar to that of differentiation. We illustrate the method by means of the following examples :

**Ex. 3.** *Evaluate*

(*i*) $\int_0^\infty e^{-x^2}dx$ **[Meerut 71]**

(*ii*) $\int \exp\left\{-\left(x^2+\frac{\alpha^2}{x^2}\right)\beta^2\right\}dx$

(*iii*) $\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx$.

**Sol.** (i) Let $I=\int_0^\infty e^{-x^2}dx$.

Putting $x=\alpha y$, we have

$$I=\int_0^\infty \alpha.e^{-\alpha^2y^2}dy.$$

Multiplying both sides by $e^{-\alpha^2}$, we have

$$Ie^{-\alpha^2}=\int_0^\infty \alpha.\exp\{-\alpha^2(1+y^2)\}\,dy$$

Now integrating both sides w.r.t. $\alpha$ between the limits 0 and $\infty$, we have

$$I\int_0^\infty e^{-\alpha^2}d\alpha=\int_0^\infty\int_0^\infty \alpha.\exp\{-\alpha^2(1+y^2)\}\,dy\,d\alpha$$

or $$I^2=\int_0^\infty\left[\frac{\exp\{-\alpha^2(1+y^2)\}}{-2(1+y^2)}\right]_0^\infty dy$$

$$=\tfrac{1}{2}\int_0^\infty\frac{1}{1+y^2}dy=\tfrac{1}{2}\left[\tan^{-1}y\right]_0^\infty=\frac{\pi}{4}$$

or $$I=\frac{\sqrt{\pi}}{2}.$$

Also from (1), we have

$$\int_0^\infty e^{-\alpha^2y^2}\,dy=I=\frac{\sqrt{\pi}}{2\alpha}.$$

(ii) Let $I=\int_0^\infty \exp\left\{-\beta^2\left(x^2+\frac{\alpha^2}{x^2}\right)\right\}dx.$

Then $$\frac{dI}{d\alpha}=-2\int_0^\infty \exp\left\{-\beta^2\left(x^2+\frac{\alpha^2}{x^2}\right)\right\}\frac{\alpha\beta^2}{x^2}\,dx.$$

Putting $\frac{\alpha}{x}=z$, we get

$$\frac{dI}{d\alpha}=-2\beta^2\int_0^\infty \exp\left\{-\left(z^2+\frac{\alpha^2}{z^2}\right)\beta^2\right\}dz=-2I\beta^2$$

or $$\int\frac{dI}{I}=-2\beta^2\int d\alpha$$

or $$\log I=-2\alpha\beta^2=\log c$$

or $$I=-ce^{-2\alpha\beta^2}.$$

To determine $c$, let $\alpha=0$ ; then

$$I=\int_0^\infty e^{-\beta^2x^2}\,dx=\frac{\sqrt{\pi}}{2\beta}.$$

$$\therefore\quad \frac{\sqrt{\pi}}{2\beta}=c.$$

Hence $$\int_0^\infty \exp\left\{-\left(x^2+\frac{\alpha^2}{x^2}\right)\beta^2\right\}dx=\frac{\sqrt{\pi}}{2\beta}\,e^{-2\alpha\beta^2}.$$

**Note.** If $\beta=1$, we have

$$\int_0^\infty \exp\left\{-\left(x^2+\frac{\alpha^2}{x^2}\right)\right\}dx-\frac{\sqrt{\pi}}{2}\,e^{-2\alpha}.$$

(iii) Let $I=\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx.$

Then $$\frac{dI}{db}=-2\int_0^\infty e^{-a^2x^2}\,x\sin 2bx\,dx.$$

$$=\left[\frac{e^{-a^2x^2}\sin 2bx}{a^2}\right]_0^\infty-\frac{2bx}{a^2}\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx$$

$$=-\frac{2b}{a^2}I,$$

or $$\int\frac{dI}{I}=-\int\frac{2b}{a^2}\,db \quad\text{or}\quad \log I=-\frac{b^2}{a^2}+\log c$$

or $$I=c\exp\left(-\frac{b^2}{a^2}\right).$$

when $b=0$, $I=\int_0^\infty e^{-a^2x^2}\,dx=\frac{\sqrt{\pi}}{2a}$, $\therefore$ $c=\frac{\sqrt{\pi}}{2a}$.

Hence $\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx=\frac{\sqrt{\pi}}{2a}\exp\left(-\frac{b^2}{a^2}\right)$.

**Ex. 4.** *Evaluate the integral.*

$$\int_0^\infty \frac{\sin mx}{a^2+x^2}\,dx$$

*and deduce the values of integrals.*

$$\int_0^\infty \frac{x\sin mx}{a^2+x^2}\,dx \text{ and } \int_0^\infty \frac{\sin mx}{x\,(a^2+x^2)}\,dx.$$

**Sol.** Let $I=\int_0^\infty \frac{\cos mx}{a^2+x^2}\,dx$

We have $\int_0^\infty 2z\exp\{-(a^2+x^2)\,z^2\}\,dz=\frac{1}{a^2+x^2}$

Now multiplying both sides by $\cos mx$ and integrating from 0 to $\infty$ w.r.t. $x$, we have

$$I=\int_0^\infty \frac{\cos mx}{a^2+x^2}\,dx$$

$$=\int_0^\infty\int_0^\infty [\cos mx.2z\exp\{-(a^2+x^2)\,z^2\}]\,dz\,dx$$

$$=\int_0^\infty 2ze^{-a^2z^2}\left[\int_0^\infty e^{-x^2z^2}\cos mx\,dx\right]dz$$

$$=\int_0^\infty 2ze^{-a^2z^2}.\frac{\sqrt{\pi}}{z}\exp\left(-\frac{m^2}{4z^2}\right)dx$$

$$=\sqrt{\pi}\int_0^\infty \exp\left(-a^2z^2-\frac{m^2}{4z^2}\right)dz$$

$$=\sqrt{\pi}\int_0^\infty \exp\left\{-a^2\left(z^2+\frac{m^2}{4a^2z^2}\right)\right\}dz$$

$$=\sqrt{\pi}.\frac{\sqrt{\pi}}{2a}\exp\left(-2.\frac{m}{2a}a^2\right)$$

Hence $\int_0^\infty \frac{\cos mx}{a^2+x^2}\,dx=\frac{\pi}{2a}e^{-ma}$. ...(1)

Differentiating (1) w.r.t. $m$, we have

$$\int_0^\infty \frac{x\sin mx}{a^2+x^2}\,dx=\frac{\pi}{2}e^{-am} \qquad ...(2)$$

Again integrating (1) w.r.t. $m$ between limits 0 and $m$, we have

$$\int_0^\infty \frac{\sin mx}{x\,(a^2+x^2)}\,dx=\frac{\pi}{2a^2}\,(1-e^{-ma}), \quad \ldots(3)$$

Note. If $a=1$, we have

$$\int_0^\infty \frac{\cos mx}{1+x^2}\,dx=\frac{\pi}{2}e^{-m},$$

$$\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx=\frac{\pi}{2}\,e^{-m},$$

$$\int_0^\infty \frac{\sin mx}{x\,(1+x^2)}\,dx=\frac{\pi}{2}\,(1-e^{-m}).$$

## § 2·9. Euler's Integrals—Bata and Gamma Functions.

**[Kanpur 83, 84, 86, 87; Meerut 1972, 73, 78, 79, 79, (S), 81, 88.]**

We define the first and second Eulerian integrals as

$$B\,(l, m)=\int_0^1 x^{l-1}\,(1-x)^{m-1}\,dx$$

and

$$\Gamma\,(n)=\int_0^\infty e^{-x}\,x^{n-1}\,dx,$$

and refer to them as the **Beta and Gamma** Functions respectively. These integrals occupy a vary important place in definite integrals and have got wide applications in Mechanics, Physics, Statistics and many other applied sciences.

We shall assume that all the quantities $l, m, n,$ are positive but not necessarily integrals.

It will be noted that **Beta Function** is symmetrical in $l$ and $m$, *i. e.*

$$B\,(l\ m)=B\,(m, l)$$

For

$$B\,(l, m)=\int_0^1 x^{l-1}\,(1-x)^{m-1}\,dx$$

$$=\int_0^1 (1-x)^{l-1}\,[1-(1-x)]^{m-1}\,dx$$

$$=\int_0^1 x^{m-1}(1-x)^{l-1}\,dx=B\,(m, l).$$

## § 2·10. Elementary Properties of Gamma Function.

We have

$$\Gamma\,(n)=\int_0^\infty x^{m-1}\,e^{-x}\,dx$$

$$=\Big[-x^{m+1}\,e^{-x}\Big]_0^\infty+(n-1)\int_0^\infty e^{-x}\,x^{n-2}\,dx.$$

Now $\lim_{x \to 0} x^{n-1} e^{-x} = \lim_{x \to 0} \frac{x^{n-1}}{e^x}$

$$= \lim_{x \to 0} \frac{x^{n-1}}{1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots\ldots} = 0.$$

Hence $\Gamma(n) = (n-1) \int_0^\infty e^{-x} x^{n-2} \, dx = (n-1) \Gamma(n-1)$.

Similarly $\Gamma(n-1) = (n-2) \Gamma(n-2)$ and on.

If follows that when $n$ is a positive integer, we have

$$\Gamma(n) = (n-1)(n-2) \ldots 3.2.1. \Gamma(1).,$$

and
$$\Gamma(1) = \int_0^\infty e^{-x} \, dx = \Big[ -e^{-x} \Big]_0^\infty = 1.$$

It may be further remarked that

$$\Gamma(0) = \infty \text{ and } \Gamma(-n) = \infty$$

when $n$ is a positive integer.

To sum up,

$\Gamma(n) = (n-1)!$. when $n$ is a positive integer.

$\Gamma(n) = (n-1) \Gamma(n-1)$ for all values of $n$,

$\Gamma(1) = 1$, $\Gamma(0) = \infty$,

$\Gamma(-n) = \infty$, when $n$ ia positive integer.

**§ 2·11. Transformation of Gamma Function.**

(i) Putting $x = \log \frac{1}{y}$ or $y = e^{-x}$, we have

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx = -\int_1^0 \left( \log \frac{1}{y} \right)^{n-1} dy$$

$$= \int_0^1 \left( \log \frac{1}{y} \right)^{n-1} dy.$$ [Meerut 86]

(ii) If we write $x = ky$, we have

$$\Gamma(n) = \int_0^\infty e^{-yk} k^n y^{n-1} \, dy = k^n \int_0^\infty e^{-yk} y^{n-1} \, dy$$

whence
$$\int_0^\infty e^{-ky} y^{n-1} \, dy = \frac{\Gamma(n)}{k^n}.$$

(iii) Again if we put $x^n = y$, we have

$$\Gamma(n) = \frac{1}{n} \int_0^\infty e^{-(y)^{1/n}} \, dy$$

or
$$\int_0^\infty \exp.\{-(y)^{1/n}\} \, dy = n \, \Gamma(n) = \Gamma(n+1).$$

Putting $n=\frac{1}{2}$, we get

$$\int_0^{\infty} e^{-y^2} dy = \frac{1}{2}\,\Gamma(\tfrac{1}{2}).$$

**Illustrative Examples**

**Ex. 5.** *Compute* (*i*) $\Gamma(-1/2)$, (*ii*) $\Gamma(-5/2)$.

**Sol.** (i) We know that $\Gamma(n+1)= \Gamma(n)$.

$$\therefore\ \Gamma(n)=\frac{\Gamma(n+1)}{n}. \qquad \ldots(1)$$

Putting $n=-\frac{1}{2}$, we get

$$\Gamma(-\tfrac{1}{2})=\frac{\Gamma(-\frac{1}{2}+1)}{(-\frac{1}{2})}=\frac{\Gamma(\frac{1}{2})}{-(\frac{1}{2})}=-2\sqrt{\pi}. \qquad (\because \Gamma(\tfrac{1}{2})=\sqrt{\pi})$$

(ii) Now Putting $n=-5/2$ in (1), we get

$$\Gamma(-5/2)=\frac{\Gamma(-5/2+1)}{(-5/2)}=-\tfrac{2}{5}\,\Gamma(-3/2) \qquad \ldots(2)$$

Aagin putting $n=-3/2$ in (1), we get

$$\Gamma(-3/2)=\frac{\Gamma(-3/2+1)}{(-3/2)}=-\tfrac{2}{3}\,\Gamma(-\tfrac{1}{2})=-\tfrac{2}{3}\,(-2\sqrt{\pi})$$

[from (i)]

$$=\tfrac{4}{3}\sqrt{\pi}.$$

$\therefore$ From (2), we have

$$\Gamma(-5/2)=(-\tfrac{2}{5}).(\tfrac{4}{3}\sqrt{\pi})=-\tfrac{8}{15}\sqrt{\pi}.$$

**Ex. 6.** *When n is a positive integer, prove that*

$$\Gamma(-n+)=\frac{(-!)^n\,2^n\sqrt{\pi}}{1.3.5\ldots(2n-1)}$$

**Sol.** we have

$$\Gamma(n+1)=n\Gamma(n)\ ;\ \Gamma(n)=\frac{\Gamma(n+1)}{n}. \qquad \ldots(1)$$

$$\therefore\ \Gamma(-n+\tfrac{1}{2})=\Gamma\left(\frac{-2n+1}{2}\right)=\frac{\Gamma\left(\frac{-2n+1}{2}+1\right)}{\left(\frac{-2n+1}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{-2n+3}{2}\right)}{\left(\frac{-2n+1}{2}\right)}$$

$$=\frac{1}{\left(\frac{-2n+1}{2}\right)}\cdot\frac{\Gamma\left(\frac{-2n+3}{2}+1\right)}{\left(\frac{-2n+3}{2}\right)}$$

[Using (1) again]

$$=\frac{\Gamma\left(\frac{-2n+5}{2}\right)}{\left(\frac{-2n+1}{2}\right)\left(\frac{-2n+3}{2}\right)}$$

Proceeding similarly again and again, we get

$$\Gamma(-n+\tfrac{1}{2})=\frac{1}{\left(\frac{-2n+1}{2}\right)}\cdot\frac{1}{\left(\frac{-2n+3}{2}\right)}\cdot\frac{1}{\left(\frac{-2n+5}{2}\right)}\cdots\cdots\frac{1}{(-\frac{3}{2})}\cdot\frac{\Gamma(-\frac{1}{2}+1)}{(-\frac{1}{2})}$$

$$=\frac{2^n}{(-2n+1)(-2n+3)(-2n+5)\ldots(-3)(-1)}\Gamma(\tfrac{1}{2})$$

$$=\frac{2^n}{(-1)^n(2n-1)(2n-3)\ldots3.1}\sqrt{\pi}$$

$$=\frac{(-1)^n\,2^n\sqrt{\pi}}{(-1)^{2n}.1.3.5\ldots(2n-1)}=\frac{(-1)^n\,2^n\sqrt{\pi}}{1.3.5\ldots(2n-1)}.$$

**Ex. 7.** *Show that*

$$2^n\,\Gamma(n+\tfrac{1}{2})=1.3.5\ldots(2n-1)\sqrt{\pi}.$$

*where n is a positive integer.*

**Sol.** We have

$$\Gamma(n+\tfrac{1}{2})=\Gamma\{(n-\tfrac{1}{2})+1\}=(n-\tfrac{1}{2})\,\Gamma\left\{n-\frac{1}{2}\right\}$$

$$\Gamma(n+\tfrac{1}{2})=(n-\tfrac{1}{2})\,\Gamma\left(n-\frac{1}{2}\right)$$

$$=(n-\tfrac{1}{2}).(n-\tfrac{3}{2})\,\Gamma\left\{n-\frac{3}{2}\right\}$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$=(n-\tfrac{1}{2})(n-\tfrac{3}{2})(n-\tfrac{5}{2})\ldots\tfrac{3}{2}.\tfrac{1}{2}.\Gamma(\tfrac{1}{2})$$

$$=\frac{(2n-1)(2n-3)(2n-5)\ldots3.1}{2^n}.\sqrt{\pi}$$

or $$2^n\,\Gamma(n+\tfrac{1}{2})=1.3.5\ldots(2n-1)\sqrt{\pi}.$$

**Ex. 8.** *Evaluate (i)* $\int_0^\infty x^4 e^{-x}\,dx$, *(ii)* $\int_0^\infty x^2 e^{-x^2}\,dx$

*(iii)* $\int_0^\infty \sqrt{x}.e.^{-x^3}\,dx$ *(iv)* $\int_0^\infty t^{-3/2}(1-e^{-t})\,dt.$

**Sol.** (i) We have, $\int_0^\infty x^4 e^{-x}\,dx=\int_0^\infty e^{-x}\,x^{5-1}\,dx$

$=\Gamma(5)$ $\left[\because \int_0^\infty e^{-x} x^{n-1}\, dx=\Gamma(n)\right]$

$=4\,!=24$ $[\because \Gamma(n+1)=n\,!$ where $n$ is positive integer$]$

(ii) Putting $x^2=t$ *i.e.* $x=t^{1/2}$, so that $dx=\frac{1}{2}t^{-1/2}$, we get

$$\int_0^\infty x^3 e^{-x^2} dx=\int_0^\infty t\, e^{-t}.\,(\tfrac{1}{2}t^{-1/2})\, dt.$$

$$=\tfrac{1}{2}\int_0^\infty e^{-t}.t^{-\frac{1}{2}+1} dt=\tfrac{1}{2}\int_0^\infty e^{-t}.t^{\frac{3}{2}-1}\, dt$$

$$=\tfrac{1}{2}\,\Gamma(\tfrac{3}{2})=\tfrac{1}{2}\,\Gamma(\tfrac{1}{2}+1)=\tfrac{1}{2}.\tfrac{1}{2}\,\Gamma(\tfrac{1}{2})=\tfrac{1}{4}\sqrt{\pi}. \qquad [\because \Gamma(\tfrac{1}{2})=\sqrt{\pi}]$$

(iii) Putting $x^3=t$, *i. e.* $x=t^{1/3}$, so that $dx=\frac{1}{3}t^{-2/3}\, dt$, we get

$$\int_0^\infty \sqrt{x}.e^{-x^3} dx=\int_0^\infty (t^{1/3})^{1/2}.e^{-t}.(\tfrac{1}{3}t^{-2/3})\, dt$$

$$=\tfrac{1}{3}\int_0^\infty e^{-t}\, t^{-1/2}\, dt=\tfrac{1}{3}\int_0^\infty e^{-t}\, t^{\frac{1}{2}-1} dt$$

$$=\tfrac{1}{3}\,\Gamma(\tfrac{1}{2})=\tfrac{1}{3}\sqrt{\pi}.$$

(iv) Integrating by parts taking $1-e^{-t}$ as the first function,

$$\int_0^\infty t^{-3/2}(1-e^{-t})\, dt=\Big[(1-e^{-t})\ (-2t^{-1/2})\Big]_0^\infty-\int_0^\infty e^{-t}\,(-2t^{-1/2})dt$$

$$=0+2\int_0^\infty e^{-t}.t^{-1/2}\ dt=2\int_0^\infty e^{-t}\, t^{\frac{1}{2}-1}\, dt=2\ \Gamma(\tfrac{1}{2})=2\sqrt{\pi}.$$

**Ex. 9.** *Evaluate* $\int_0^1 \frac{dx}{\sqrt{(-\log x)}}$.

**Sol.** Putting $-\log x=t$ *i.e.* $x=e^{-t}$, so that $dx=-e^{-t}\, dt$, we get

$$\int_0^1 \frac{dx}{\sqrt{(-\log x)}}=\int_\infty^0 \frac{-e^{-t}\, dt}{\sqrt{t}}=\int_0^\infty e^{-t}\, t^{\frac{1}{2}-1}\, dt$$

$$=\Gamma(\tfrac{1}{2})=\sqrt{\pi}.$$

**Ex. 10-** *prove that, if $n$ is a positive integer and $m>-1$,*

$$\int_0^1 x^m(\log x)^n\, dx=\frac{(-1)^n\, n\,!}{(m+1)^{n+1}} \qquad \text{[Meerut 74]}$$

**Sol.** Putting $\log x=-t$ *i.e.* $x=e^{-t}$ and $dx=-e^{-t}\, dt$.

$$\int_0^1 (\log x)^n\, dx=\int_\infty^0 (e^{-t})^m\, (-t)^n\, (-e^{-t})\, dt.$$

$$=(-1)^n\int_0^\infty e^{-(m+1)t}.t^{(n+1)-1}\, dt$$

$$=(-1)^n\frac{\Gamma(n+1)}{(m+1)^{n+1}} \qquad [\text{see § 2·11 (i)}]$$

$$=\frac{(-1)^n\, n\,!}{(m+1)^{n+1}}.$$

**Ex. 11.** *Show that*

$$\int_0^\infty x^{2n-1} e^{-ax^2} dx = \frac{\Gamma(n)}{2a^n}, a > 0, n > 0. \quad \text{[Agra 83]}$$

**Sol. Putting.** $-ax^2 = t$ *i.e.* $x = -\frac{1}{\sqrt{a}} t^{1/2}$

and $dx = -\frac{1}{2a} t^{-1/2} dt$, we get

$$\int_0^\infty x^{2n-1} e^{-ax^2} dx = \int_0^\infty \left( -\frac{1}{\sqrt{a}} t^{1/2} \right)^{2n-1} e^{-t} \left( -\frac{1}{2a} t^{-1/2} \right) dt$$

$$= \frac{1}{2a^n} \int_0^\infty e^{-t}\, t^{n-1}\, dt = \frac{1}{2a^n} \Gamma(n)$$

**Illustrative Examples**

**2·12. Another form of Beta Function.**

If we substitute $\frac{1}{1+y}$ for $x$, we get

$$B(l, m) = \int_0^1 x^{l-1} (1-x)^{m-1}\, dx \Big\}.$$

$$= \int_\infty^0 \frac{1}{(1+y)^{l-1}} \cdot \frac{y^{m-1}}{(1+y)^{m-1}} \left\{ -\frac{1}{(1+y)^2} \right\} dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{l+m}} dy.$$

Again since Beta Function is symmetrical in $l$, m we also have

$$B(l, m) = \int_0^\infty \frac{y^{l-1}}{(1+y)^{l+m}} dy.$$

which can also be obtained by putting $x = \frac{y}{1+y}$ in the original form.

**§ 2·13. Relation between Beta and Gamma Functions.**

To prove that

$$B(l, m) = \frac{\Gamma(l)\, \Gamma(m)}{\Gamma(l+m)}$$

**[Bombay 70; Meerut 72, 73, 77, 78, 79 (S), 81, 82, 84 (P), 86, 87, 88: Agra 72, 74, 76, 84, 86: Kanpur 83, 86, 87]**

**Proof.** We have

$$\frac{\Gamma(l)}{z^l} = \int_0^\infty e^{-zx}\, x^{l-1}\, dx$$

or

$$\Gamma(l) = \int_0^\infty z^l e^{-zx} \cdot x^{l-1}\, dx.$$

Multiplying both sides by $e^{-z} z^{m-1}$, we have

$$\Gamma(l)\, e^{-z}\, z^{m-1} = \int_0^\infty e^{-z(1+x)}\, z^{l+m-1} \cdot x^{l-1}\, dx.$$

Now integrating both sides w.r.t. $z$ from 0 to $\infty$, we have

$$\Gamma(l)\int_0^\infty e^{-z}.z^{m-1}\,dz=\int_0^\infty\left[\int_0^\infty e^{-z(1+x)}.z^{l+m-1}\,dz\right]x^{l-1}\,dx$$

or
$$\Gamma(l)\ \Gamma(m)=\int_0^\infty \frac{\Gamma\ (l+m)}{(1+x)^{l+m}}.x^{l-1}\,dx \qquad ...(1)$$
$$=\Gamma\ (l+m)\ B\ (l,\ m).$$

Hence
$$B\ (l,\ m)=\frac{\Gamma(l)\,\Gamma(m).}{\Gamma\ (l+m)}$$

**Deductions.** 1. Putting $l+m=1$ in (1), we have

$$\Gamma(l)\ \Gamma\ (1-l)=\int_0^\infty \frac{x^{l-1}}{1+x}\,dx=\frac{\pi}{\sin l\pi} \qquad ...(2)$$

$$\therefore \int_0^\infty \frac{x^{l-1}}{1+x}\,dx=\frac{\pi}{\sin l\pi} \text{ and } \Gamma\ (1)=1.$$

2. Again if we put $l=\frac{1}{2}$ in (2), we have

$$[\Gamma(\tfrac{1}{2})]^2=\pi$$
$$\Gamma(\tfrac{1}{2})=\sqrt{\pi}$$

§ 2·14. **Other Transformations.** Beta function can be transformed into many other forms. A few of them are given below.

(i) We have

$$\int_0^\infty \frac{y^{l-1}}{(1+y)^{l+m}}\,dy=\int_0^1 \frac{y^{l-1}}{(1+y)^{l+m}}\,dy+\int_0^\infty \frac{y^{l-1}}{(1+y)^{l+m}}\,dx$$

Now, substituting $1/x$ for $y$ in the last integral, we get

$$\int_0^\infty \frac{y^{l-1}}{(1+y)^{l+m}}\,dy=\int_0^1 \frac{x^{m-1}\,dx}{(1+x)^{l-m}}$$

Hence
$$\int_0^\infty \frac{\mathrm{y}^{l-1}}{(1+\mathrm{y})^{l+m}}\,\mathrm{dy}=\int_0^1 \frac{\mathrm{x}^{l-1}+\mathrm{x}^{m-1}}{(1+\mathrm{x})^{l+m}}\,\mathrm{dx}$$
$$=\frac{\Gamma\ (l)\ \Gamma\ (m)}{\Gamma\ (l+m)} \qquad \text{(from § 2·12)}$$

i.e
$$\int_0^1 \frac{\mathrm{x}^{l-1}+\mathrm{x}^{m-1}}{(1+\mathrm{x})^{l+m}}\,\mathrm{dx}=\int_0^\infty \frac{\mathrm{x}^{l-1}}{(1+\mathrm{x})^{l+m}}\,\mathrm{dx}=\mathrm{B}\ (l,\ \mathrm{m})$$

(ii) If we put $x=\frac{ay}{b}$, we get

$$\int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\,dx=a^l b^m\int_0^\infty \frac{y^{l-1}}{(ay+b)^{l+m}}\,dy$$

Hence
$$\int_0^\infty \frac{y^{l-1}}{(ay+b)^{l+m}}\,dy=\frac{\Gamma\ (l).\Gamma\ (m)}{a^l.b^m\ \Gamma\ (l+m)}.$$

Again putting $y=\tan^2\theta$ in this, we get

$$\int_0^{\pi/2} \frac{\sin^{2l-1}\theta\ \cos^{2m-1}\theta\ d\theta}{(a\sin^2\theta+b\cos^2\theta)^{l+m}}=\frac{\Gamma\ (l).\Gamma\ (m)}{2a^l.b^m.\Gamma\ (l+m)}$$

(iii) Let $x=\sin^2\theta$; then we get

$$\int_0^1 x^{l-1}(1-x)^{m-1}\,dx=2\int_0^{\pi/2}\sin^{2l-1}\theta\cos^{2m-1}\theta\,d\theta.$$

$$\therefore\int_0^{\pi/2}\sin^{2l-1}\theta\cos^{2m-1}\theta\,d\theta=\frac{\Gamma(l).\Gamma(m)}{2\Gamma(l+m)}.$$

This result may also be written in the form

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

(i) Put $y=\dfrac{x-b}{a-b}$ so that $dy=\dfrac{1}{a-b}dx$.

Then

$$B(l,m)=\int_0^1 y^{l-1}(1-y)^{m-1}\,dy=\int_b^a\left(\frac{x-b}{a-b}\right)^{l-1}\left(\frac{a-x}{a-b}\right)^{m-1}\frac{dx}{a-b}$$

$$=\frac{1}{(a-b)^{l+m-1}}\cdot\int_b^a(x-b)^{l-1}(a-x)^{m-1}\,dx$$

or

$$\int_b^a(x-b)^{l-1}(a-x)^{m-1}\,dx=(a-b)^{l+m-1}B(l,m)$$

$$=(a-b)^{l+m-1}\frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)}$$

**Illustrative Examples**

**Ex. 12.** *Evaluate* (i) $\int_0^1 x^4(1-x)^2\,dx$

(ii) $\int_0^a x^4\sqrt{(a^2-x^2)}\,dx$

(iii) $\int_0^2 x(8-x^3)^{1/2}\,dx.$ [Meerut 77, 75(S)]

**Sol.** (i) $\int_0^1 x^4(1-x)^2\,dx=\int_0^1 x^{5-1}(1-x)^{3-1}\,dx$

$$=B(5,3)=\frac{\Gamma(5)\,\Gamma(3)}{\Gamma(5+3)}=\frac{4!.\,2!}{7!}=\frac{4!.2.!}{7.6.5.4!}=\frac{1}{105}.$$

(ii) **Putting** $x^2=a^2t$, *i.e.*, $x=at^{1/2}$

and $dx=\frac{1}{2}at^{-1/2}\,dt$, we have

$$\int_0^a x^4\sqrt{(a^2-x^2)}\,dx=\int_0^1 a^4t^2.\sqrt{(a^2-a^2t)}.\tfrac{1}{2}at^{-1/2}\,dt$$

$$=\tfrac{1}{2}a^6\int_0^1 t^{5/2-1}.(1-t)^{3/2-1}\,dt=\tfrac{1}{2}a^6\,B(5/2,3/2)$$

$$=\tfrac{1}{2}a^6 \frac{\Gamma(5/2)\,\Gamma(3/2)}{\Gamma(5/2,\,3/2)}=\tfrac{1}{2}a^6 \cdot\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{3\,!}=\frac{\pi a^6}{32}$$

(iii) Putting $x^3=8t$ i.e. $x=2t^{1/3}$

and $dx=\frac{2}{3}t^{-2/3}\,dt$, we have

$$\int_0^2 x\,(8-x^3)^{1/3}\,dx=\int_0^1 2t^{1/3}.(8-8t)^{1/3}.\tfrac{2}{3}t^{-2/3}\,dt$$

$$=\frac{8}{3}\int_0^1 t^{-1/3}\,(1-t)^{1/3}\,dt=\frac{8}{3}\int_0^1 t^{2/3-1}.(1-t)^{4/3-1}\,dt.$$

$$=\frac{8}{3}\,B\left(\tfrac{2}{3},\tfrac{4}{3}\right)=\frac{8}{3}\,\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)}$$

$$=\frac{8}{3}\cdot\frac{\Gamma\left(\frac{2}{3}\right).\frac{1}{3}\Gamma\left(\frac{1}{3}\right)}{\Gamma(2)}=\frac{8}{9}.\,\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(1-\tfrac{1}{3}\right)$$

$$=\frac{8}{9}\cdot\frac{\pi}{\sin(\pi/3)}. \qquad (\because\ \Gamma(x)\,\Gamma(1-x)=\pi/\sin x\pi)$$

$$=\frac{16\pi}{9\sqrt{3}}.$$

**Ex. 13.** *Evaluate*

(i) $\displaystyle\int_0^2 \frac{x^2\,dx}{\sqrt{(2-x)}}$ (ii) $\displaystyle\int_0^\infty \frac{x^8\,(1-x^6)}{(1+x)^{24}}\,dx$ (iii) $\displaystyle\int_0^\infty \frac{x^4\,(1+x^6)}{(1+x)^{15}}\,dx.$

**Sol.** (i) $\displaystyle\int_0^2 \frac{xt^2\,dx}{\sqrt{(2-x)}}$

$$=\int_0^1 \frac{4t^2.2dt}{\sqrt{(2-2t)}} \qquad \text{Putting } x=2t \text{ and } dx=2dt$$

$$=4\sqrt{2}\int_0^1 t^2\,(1-t)^{-1/2}\,dt=4\sqrt{2}\int_0^1 t^{3-1}\,(1-t)^{1/2-1}\,dt$$

$$=4\sqrt{2}\,B\left(3,\tfrac{1}{2}\right)=4\sqrt{2}\,\frac{\Gamma(3)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(3+\frac{1}{2}\right)}$$

$$=4\sqrt{2}.\,\frac{2.1.\sqrt{\pi}}{\Gamma\left(\frac{7}{3}\right)}=4\sqrt{2}.\,\frac{2.1.\sqrt{\pi}}{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}=\frac{64\sqrt{2}}{15}.$$

(ii) $$\int_0^\infty \frac{x^8\,(1-x^6)}{(1+x)^{24}}\,dx=\int_0^\infty \frac{x^8-x^{14}}{(1+x)^{24}}\,dx$$

$$=\int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx-\int_0^\infty \frac{x^{14}\,dx}{(1+x)^{24}}$$

$$=\int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx-\int_0^\infty \frac{x^{15-1}\,dx}{(1+x)^{15+9}}$$

$$=B(9,15)-B(15,9)=B(9,15)-B(9,15)=0.$$

(iii) $$\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx=\int_0^\infty \frac{x^4+x^9}{(1+x)^{15}}\,dx$$

$$=\int_0^\infty \frac{x^4\,dx}{(1+x)^{15}}+\int_0^\infty \frac{x^9\,dx}{(1+x)^{15}}$$

$$=\int_0^\infty \frac{x^{5-1}\,dx}{(1+x)^{5+10}}+\int_0^\infty \frac{x^{10-1}\,dx}{(1+x)^{10+5}}$$

$$=B(5,10)+B(10,5)=B(5,10)+B(5,10)$$

$$=2B(5,10)=2.\frac{\Gamma(5)\,\Gamma(10)}{\Gamma(5+10)}$$

$$=\frac{2.4!.9!}{15!}=\frac{2.4.3.2.1.9!}{14.13.12.11.10.9!}=\frac{1}{5005}.$$

**Ex. 14.** *Show by means of Beta function that*

$$\int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^\alpha}=\frac{\pi}{\sin\pi\alpha},(0<\alpha<1).$$

[Rohilkhand 76]

**Sol.** Putting $x-t=(z-t)\,y$ i.e. $x=t+(z-t)\,y$

and $dx=(z-t)\,dy$,

we have (Note that $z$ and $t$ are constants)

$$\int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^\alpha}=\int_0^1 \frac{(z-t)\,dy}{\{z-t-(z-t)\,y\}^{1-\alpha}\,\{(z-t)y\}^\alpha}$$

$$=\int_0^1 \frac{(z-t)\,dy}{\{(z-t)\,(1-y)\}^{1-\alpha}\,(z-t)^\alpha.y^\alpha}$$

$$=\int_0^1 \frac{dy}{(1-y)^{1-\alpha}.\,y^\alpha}=\int_0^1 y^{-\alpha}(1-y)^{\alpha-1}\,dy$$

$$=\int_0^1 y^{(1-\alpha)-1}(1-y)^{\alpha-1}\,dy=B(1-\alpha,\alpha)$$

$$=\frac{\Gamma(1-\alpha)\,\Gamma(\alpha)}{\Gamma(1-\alpha+\alpha)}=\Gamma(1-\alpha)\,\Gamma(\alpha)=\frac{\pi}{\sin\pi\alpha}.$$

**Ex. 15.** *Show that*

(i) $\Gamma(\frac{3}{2}-x)\,\Gamma(\frac{3}{2}+x)=(\frac{1}{4}-x^2)\,\pi\sec\pi x$ *provided* $-1<2x<1$.

[Agra 76]

(ii) $\Gamma(\frac{1}{4}).\Gamma(\frac{3}{4})=\pi\sqrt{2}.$

**Sol.** (i) $\Gamma(\frac{3}{2}-x)\,\Gamma(\frac{3}{2}+x)=(\frac{1}{2}-x)\,\Gamma(\frac{1}{2}-x)\,(\frac{1}{2}+x)\,\Gamma(\frac{1}{2}+x)$

$$=(\tfrac{1}{4}-x^2)\,\Gamma(\tfrac{1}{2}-x)\,\Gamma(\tfrac{1}{2}+x)=(\tfrac{1}{4}-x^2)\,\Gamma\left(\frac{1-2x}{2}\right)\Gamma\left(\frac{1+2x}{2}\right)$$

$$=(\tfrac{1}{4}-x^2)\,\Gamma\left(\frac{1-2x}{2}\right)\Gamma\left(1-\frac{1-2x}{2}\right)$$

$$=(\tfrac{1}{4}-x^2).\frac{\pi}{\sin\left\{\pi\left(\frac{1-2x}{2}\right)\right\}}=(\tfrac{1}{4}-x^2).\frac{\pi}{\sin\left(\frac{\pi}{2}-\pi x\right)}$$

$=(\frac{1}{4}-x^2)\,\pi/\cos \pi x=(\frac{1}{4}+x^2)\,\pi \sec \pi\, x.$

(ii) $\Gamma(\frac{1}{4})\,\Gamma(\frac{3}{4})=\Gamma(\frac{1}{4}).\Gamma\left(1-\frac{1}{4}\right)=\dfrac{\pi}{\sin(\pi.\frac{1}{4})}$

$$=\frac{\pi}{(1/\sqrt{2})}=\sqrt{2}\,\pi.$$

**Ex. 16.** *Show that*

(*i*) $\Gamma(x)\,\Gamma(-x)=-\dfrac{\pi}{x \sin \pi x}$

(*ii*) $\Gamma(\frac{1}{2}+x)\,\Gamma(\frac{1}{2}-x)=\dfrac{\pi}{\cos \pi x}.$

**Sol.** (i) We have

$$\Gamma(x)\,\Gamma(1-x)=\pi/\sin \pi x. \qquad \text{...(1)}$$

Also $\Gamma(1-x)=\Gamma\{(-x)+1\}=(-x)\,\Gamma(-x).$ ...(2)

$\therefore$ from (i) and (ii), we have

$$\Gamma(x).(-x)\,\Gamma(-x)=\pi/\sin \pi x$$

or

$$\Gamma(x)\,\Gamma(-x)=-\pi/x \sin \pi\, x.$$

(ii) $\Gamma(\frac{1}{2}+x)\,\Gamma(\frac{1}{2}-x)$

$$=\Gamma\left(\frac{1+2x}{2}\right)\Gamma\left(\frac{1-2x}{2}\right)$$

$$=\Gamma\left(1-\frac{1-2x}{2}\right)\Gamma\left(\frac{1-2x}{2}\right)=\frac{\pi}{\sin\left\{\pi\left(\frac{1-2x}{2}\right)\right\}}$$

$$=\frac{\pi}{\sin\left(\frac{\pi}{2}-\pi x\right)}=\frac{\pi}{\cos \pi x}.$$

**Ex. 17.** *Show that*

$$\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}\,dx=2^{p+q-1}\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}.$$

[Meerut 83 P]

**Sol.** Putting $1+x=(1-x)\,t$ i.e., $t=\dfrac{1+x}{1-x}$, $x=\dfrac{t-1}{t+1}$, and

$$dx=\frac{2}{(t+1)^2}\,dt, \text{ we get}$$

$$\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}\,dx=\int_0^\infty \{(1-x)\,t\}^{p-1}.(1-x)^{q-1}\frac{2}{(1+t)^2}dt$$

$$=2\int_0^\infty \frac{t^{p-1}}{(1+t)^2}(1-x)^{p+q-2}\,dt=2\int_0^\infty \frac{t^{p-1}}{(1+t)^2}\left(1-\frac{t-1}{t+1}\right)^{p+q-2}dt$$

$$=2\int_0^\infty \frac{t^{p-1}}{(1+t)^2}\left(\frac{2}{t+1}\right)^{p+q-2}dt=2^{p+q-1}\int_0^\infty \frac{t^{p-1}\,dt}{(1+t)^{p+q}}$$

$=2^{p+q-1} B(p, q)$ (From § 2·14)

$=2^{p+q-1}.\dfrac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}.$

**§ 2·15. Legendre Duplication formula.**

**To prove that**

$$\Gamma(m).\,\Gamma(m+\tfrac{1}{2})=\frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$$

**where m is an integer.**

**[Delhi 78; Bombay 70; Meerut 81, 81 (P), 82 (P), 83 (P), 85; Kanpur 85; Agra 80]**

**Proof.** We have already proved that

$$\int_0^{\pi/2} \sin^{2l-1}\theta \cos^{2m-1}\theta\, d\theta=\frac{\Gamma(l)\,\Gamma(m)}{2\Gamma(l+m)} \qquad \ldots(1)$$

In this, put $l=\frac{1}{2}$

Then $$\int_0^{\pi/2} \cos^{2m-1}\theta\, d\theta=\frac{\sqrt{\pi}}{2}.\frac{\Gamma(m)}{\Gamma\left(\dfrac{2m+1}{2}\right)} \qquad \ldots(2)$$

Again putting $l=m$ in (1), we get

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta\, d\theta=\frac{[\Gamma(m)]^2}{2\Gamma(2m)}.$$

or $$\frac{[\Gamma(m)]^2}{2\Gamma(2m)}=\frac{1}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1} 2\theta\, d\theta$$

$$=\frac{1}{2^{2m}}\int_0^{\pi} \sin^{2m-1} z\, dz, \text{ where } 2\theta=z.$$

$$=\frac{1}{2^{2m}}\int_0^{\pi/2} \sin^{2m-1} z\, dz.$$

Hence

$$\frac{2^{2m-2}[\Gamma(m)]^2}{\Gamma(2m)}=\int_0^{\pi/2} \sin^{2m-1} z\, dz=\int_0^{\pi/2} \cos^{2m-1} z\, dz. \qquad \ldots(3)$$

Therefore from (2) and (3), we have

$$\frac{2^{2m-2}[\Gamma(m)]^2}{\Gamma(2m)}=\frac{\sqrt{\pi}}{2}.\frac{\Gamma(m)}{\Gamma\left(\dfrac{2m+1}{2}\right)}$$

or $$\Gamma(m)\,\Gamma(m+\tfrac{1}{2})=\frac{\sqrt{\pi}}{2^{2m-1}}.\Gamma(2m). \qquad \ldots(4)$$

**Deductions.** If $m$ is a positive integer then

$$\frac{\Gamma(2m)}{\Gamma(m)}=\frac{(2m-1)\,!}{(m-1)\,!}=\frac{2m\,(2m-1)\,!}{2m\,(m-1)\,!}=\frac{(2m)\,!}{2.m\,!}. \qquad \ldots(5)$$

∴ From (4), we get

$$\Gamma(m+\tfrac{1}{2})=\frac{\sqrt{\pi}}{2^{2m-1}}.\frac{\Gamma(2m)}{\Gamma(m)}$$ **[Kanpur 85]**

$$=\frac{\sqrt{\pi}}{2^{2m-1}}\cdot\frac{(2m)\,!}{2.m\,!} \qquad \text{from (5)}$$

or $$\Gamma\left(m+\tfrac{1}{2}\right)=\frac{(2m)\,!}{2^{2m}\,m\,!}\sqrt{\pi}.$$

**§ 2·16. To Prove that**

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}$$

**where n is an integer.** **[Meerut 83, 87; B.H.U. 70]**

**Proof.** Let $A=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)$

Writing the expression in the reserved order, we have

$$A=\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\ldots\Gamma\left(1-\frac{n-2}{2}\right)\Gamma\left(1-\frac{n-1}{n}\right)$$

Multiplying these, we get

$$A^2=\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\ldots$$
$$\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right)$$

$$=\frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\sin\ldots\frac{n-1}{n}\pi}.$$

To calculate this expression, we factorize $1-x^{2n}$.

Now the roots of the equation $x^{2n}-1=0$ are given by

$$x=(1)^{1/2n}=(\cos 2r\pi+i\sin 2r\pi)^{1/2n}$$
$$=\cos\frac{r\pi}{n}+i\sin\frac{r\pi}{n}$$

where $\quad r=0, 1, 2, \ldots, 2n-1.$

Hence we have

$$(1-x^{2n})=(1-x)(1+x)\left(x-\cos\frac{\pi}{n}-i\sin\frac{\pi}{n}\right)$$
$$\times\left(x-\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)\ldots$$
$$\ldots\times\left(x-\cos\frac{n-1}{n}\pi-i\sin\frac{n-1}{n}\pi\right)$$
$$\times\left(x-\cos\frac{n-1}{n}\pi-i\sin\frac{n-1}{n}\pi\right)$$
$$=(1-x^2)\left(1-2x\cos\frac{\pi}{n}+x^2\right)\left(1-2x\cos\frac{2\pi}{n}+x^2\right)$$
$$\ldots\left(1-2x\cos\frac{n-1}{n}\pi+x^2\right).$$

$$\therefore \frac{1-x^{2n}}{1-x^2}=\left(1-2x\cos\frac{\pi}{n}+x^2\right)\left(1-2x\cos\frac{2\pi}{n}+x^2\right)\ldots$$
$$\ldots\left(1-2x\cos\frac{n-1}{n}\pi+x^2\right)$$

Putting $x=1$ and $x=-1$ respectively, we have in the limit

$$n=\left(2-2\cos\frac{\pi}{n}\right)\left(2-2\cos\frac{2\pi}{n}\right)\ldots\left(2-2\cos\frac{n-1}{n}\pi\right).$$

and $$n=\left(2+2\cos\frac{\pi}{n}\right)\left(2+2\cos\frac{2\pi}{n}\right)\ldots\left(2+2\cos\frac{n-1}{n}\pi\right).$$

Multiplying these, we get

$$n^2=2^{2n-2}.\sin^2\frac{\pi}{n}.\sin^2\frac{2\pi}{n}\ldots\sin^2\frac{n-1}{n}\pi,$$

$$n=2^{n-1}.\sin\frac{\pi}{n}.\sin\frac{2\pi}{n}\ldots\sin\frac{n-1}{n}\pi.$$

$$\therefore \quad A^2=\frac{(2\pi)^{n-1}}{n} \quad\text{or}\quad A=\frac{(2\pi)^{(n-1)/2}}{n^{1/2}}$$

**Note.** The value of $\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{n-1}{n}\pi$ can also be found by using the identity.

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\sin\left(\theta+\frac{3\pi}{n}\right)$$
$$\ldots\sin\left(\theta+\frac{n-1}{n}\pi\right)$$

[Hobson, Trigonometry, P. 117]

For we have in the limit when $\theta=0$,

$$n=2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{n-1}{n}\pi.$$

## Miscellaneous Examples

**Ex. 18.** *Evaluate the integrals*

$$\int_0^\infty e^{-ax}x^{m-1}\cos bx\,dx,$$

*and* $$\int_0^\infty e^{-ax}.x^{m-1}\sin bx\,dx.$$

**Sol.** We have

$$\int_0^\infty e^{-x(a+bi)}.x^{m-1}\,dx=\frac{\Gamma(m)}{(a+bi)^m}. \qquad \ldots(1)$$

Now putting $a=r\cos\theta$, $b=r\sin\theta$, we have

$$\frac{\Gamma(m)}{(a+bi)^m}=\frac{\Gamma(m)}{r^m}(\cos\theta-i\sin\theta)^{-m}$$

$$=\frac{\Gamma(m)}{r^m}(\cos m\theta+i\sin m\theta)$$

where $\quad r=(a^2+b^2)^{1/2}$ and $\theta=\tan^{-1}\dfrac{b}{a}$

Hence equating real and imaginary parts in (1), we have

$$\int_0^\infty e^{-ax}.x^{m-1}\cos bx\,dx=\frac{\Gamma(m)\cos m\theta}{r^m}$$

and $$\int_0^\infty e^{-ax}.x^{m-1}\sin bx\,dx=\frac{\Gamma(m)\sin m\theta}{r^m}.$$

If we put $a=0$, $\theta$ becomes $\dfrac{\pi}{2}$. We then have

$$\int_0^\infty x^{m-1}\cos bx\,dx=\frac{\Gamma(m)}{b^m}\cos\frac{m\pi}{2}$$

and $$\int_0^\infty x^{m-1}\sin bx\,dx=\frac{\Gamma(m)}{b^m}\sin\frac{m\pi}{2}$$

**Ex. 19.** *Prove that*

$$\int_0^\infty \frac{\sin bz}{z}\,dz=\frac{\pi}{2}.$$

**Sol.** We have

$$\int_0^\infty\int_0^\infty e^{-zx}\sin bz\,dx\,dz=\int_0^\infty \frac{\sin bz}{z}\,dz.$$

Now first integrate w.r.t., $z$; then

$$\int_0^\infty\int_0^\infty e^{-zx}\sin bz\,dx\,dz$$

$$=\int_0^\infty \frac{b}{b^2+x^2}\,dx=\left[\tan^{-1}\frac{x}{b}\right]_0^\infty=\frac{\pi}{2}.$$

$$\therefore\quad \int_0^\infty \frac{\sin bz}{z}\,dz=\frac{\pi}{2}$$

**Ex. 20.** *prove that*

$$\int_0^1 \frac{x^2\,dx}{(1-x^4)^{1/2}}\times\int_0^1\frac{dx}{(1+x^4)^{1/2}}=\frac{\pi}{4\sqrt{2}}.$$ [Agra 86]

**Sol.** Putting $x^2=\sin\theta$ in the first integral, we have

$$\int_0^1 \frac{x^2\,dx}{(1-x^4)^{1/2}}=\int_0^{\pi/2}\frac{\sin\theta}{\cos\theta}.\frac{\cos\theta}{2\sqrt{(\sin\theta)}}\,d\theta$$

$$=\tfrac{1}{2}\int_0^{\pi/2}\sqrt{(\sin\theta)}\,d\theta$$

$$=\tfrac{1}{2}\cdot\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{4})}{2\Gamma(\frac{5}{4})}$$

$$=\tfrac{1}{4}\cdot\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{4})}{\frac{1}{4}\Gamma(\frac{1}{4})}=\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}$$

Again putting $x^2=\tan\phi$ in the second integral, we have

$$\int_0^1 \frac{dx}{(1+x^4)^{1/4}}=\int_0^{\pi/4}\frac{\sec^2\phi\,d\phi}{2.\sec\phi.\sqrt{(\tan\phi)}}$$

$$=\int_0^{\pi/4}\frac{d\phi}{2\sqrt{(\sin\phi.\cos\phi)}}=\frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{d\phi}{\sqrt{(\sin 2\phi)}}$$

$$=\frac{1}{2\sqrt{2}}\int_0^{\pi/4}\frac{dt}{\sqrt{(\sin t)}}\text{, where } 2\phi=t$$

$$=\frac{1}{2\sqrt{2}}\cdot\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{4})}{2\Gamma(\frac{3}{4})}.$$

Hence $\displaystyle\int_0^1\frac{x^2\,dx}{(1-x^4)^{1/2}}\times\int_0^1\frac{dx}{(1+x^4)^{1/2}}=\frac{1}{4\sqrt{2}}\{\Gamma(\tfrac{1}{2})\}^2=\frac{\pi}{4\sqrt{2}}.$

**Ex. 21.** *Evaluate*

(*i*) $\displaystyle\int_0^a\frac{dx}{(a^n-x^n)^{1/n}}.$ **[Meerut 84 (P)]**

(*ii*) $\displaystyle\int_0^1\frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx.$

**Sol.** (*i*) Put $x^n=a^n\sin^2\theta$ or $x=a\sin^{2/n}\theta$,

$\therefore\ dx=\dfrac{2a}{n}\sin^{(2-n)/n}\theta\cos\theta\,d\theta$

$$\therefore\ \int_0^a\frac{dx}{(a^n-x^n)^{1/n}}=\frac{2a}{n}\int^{\pi/2}\frac{\sin^{(2n-n)/n}\theta\cos\theta\,d\theta}{a\cos^{2/n}\theta}$$

$$=\frac{2}{n}\int_0^{\pi/2}\sin^{(2/n)-1}\theta\cos^{1-(2/n)}\theta\,d\theta$$

$$=\frac{1}{n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{2\Gamma(1)}$$

$$=\frac{1}{n}\cdot\frac{\pi}{\sin\frac{\pi}{n}}=\frac{\pi}{n\sin\frac{\pi}{n}}$$

(ii) Put $\dfrac{x(1+a)}{a+x}=y$

$\therefore\ (1+a)\,dx=dy\,(a+x)+y\,dx$

or $$dx=\frac{(a+x)\,dy}{1+a-y}$$

Also $$x=\frac{ay}{1+a-y}$$

so that $$1-x=\frac{(1+a)(1-y)}{(1+a-y)}$$

and $$a+x=\frac{a(1+a)}{1+a-y};$$

$$\therefore \int_0^1 \frac{x^{m-1}(1+x)^{n-1}}{(a+x)^{m+n}}\,dx=\int_0^1 \frac{y^{m-1}(1-y)^{n-1}}{a^n(1+a)^m}\,dy$$
$$=\frac{\Gamma(m)\,\Gamma(n)}{a^n(1+a)^m\,\Gamma(m+n)}.$$

**Ex. 22.** *Show that*

(*i*) $\displaystyle\int_0^{\pi/2} (\tan x)^n\,dx=\frac{\pi}{2}\sec\frac{n\pi}{2},\ 0<n<1.$

(*ii*) $\displaystyle\int_0^{\infty} \cos(bz^{1/n})\,dz=\frac{\Gamma(n+1)\cos\frac{n\pi}{2}}{b^n}.$

**Sol.** (i) We have

$$\int_0^{\pi/2} \tan^n x\,dx=\int_0^{\pi/2} \sin^n x\cos^{-n} x\,dx$$
$$=\frac{\Gamma\left(\frac{n+1}{n}\right)\Gamma\left(\frac{1-n}{2}\right)}{2\Gamma(1)}=\tfrac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1-\frac{n+1}{2}\right)$$
$$=\frac{1}{2}\,\frac{\pi}{\sin\frac{n+1}{2}\pi}=\frac{\pi}{2}\sec\frac{n\pi}{2}$$

(ii) Put $z=x^n$ so that $dz=nx^{n-1}\,dx$

$$\therefore \int_0^{\infty}\cos(bz^{1/n})\,dz=\int_0^{\infty} x^{n-1}\cos bx\,dx$$
$$=\text{real part of } n\int_0^{\infty} e^{-bxi}\,x^{n-1}\,dx$$
$$=\text{real part of } n\,\frac{\Gamma(n)}{(bi)^n}$$
$$=\text{real part of } \frac{\Gamma(n+1)\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^{-n}}{b^n}$$
$$=\text{real part of } \frac{\Gamma(n+1)\left(\cos\frac{\pi n}{2}+i\sin\frac{\pi n}{2}\right)}{b^n}$$

$$=\frac{\Gamma(n+1)\cos\frac{\pi n}{2}}{b^n}.$$

**Ex. 23.** *Prove that*

$$\Gamma(n)\,\Gamma\left(\frac{1-n}{2}\right)=\frac{\sqrt{\pi}.\Gamma(\frac{1}{2}n)}{2^{1-n}\cos(\frac{1}{2}n\pi)},\ 0<n<1.$$

**Sol.** We know that

$$\Gamma(m)\,\Gamma(1-m)=\pi/\sin m\pi \quad ...(1)$$

and
$$\Gamma(m)\,\Gamma(m+\tfrac{1}{2})=\frac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}} \quad ...(2)$$

(see § 2·13, and § 2·15)

Now putting $m=(n+1)/2$ in (1), we get

$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1-\frac{n+1}{2}\right)=\pi/\sin\{(n+1)\pi/2\}$$

or
$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)=\frac{\pi}{\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)}$$

or
$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)=\frac{\pi}{\cos\left(\frac{n\pi}{2}\right)} \quad ...(3)$$

Again putting $m=n/2$ in (2), we get

$$\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)=\frac{\sqrt{\pi}.\Gamma(n)}{2^{n-1}}$$

or
$$\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(n)}{2^{n-1}}$$

or
$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n)}=\frac{\sqrt{\pi}}{2^{n-1}\Gamma\left(\frac{n}{2}\right)} \quad ...(4)$$

Dividing (3) by (4), we get

$$\Gamma(n)\,\Gamma\left(\frac{1-n}{2}\right)=\frac{\sqrt{\pi}\;2^{n-1}\,\Gamma(n/2)}{\cos(\frac{1}{2}n\pi)}$$

or
$$\Gamma(n)\,\Gamma\left(\frac{1-n}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}n)}{2^{1-n}\cos(\frac{1}{2}n\pi)}.$$

**Ex. 24.** *Show that the sum of the series*

$$\frac{1}{n+1}+m\frac{1}{n+2}+\frac{m(m+1)}{2!}\frac{1}{n+3}$$

$$+\frac{m(m+1)(m+2)}{3!}\cdot\frac{1}{n+4}\cdots$$

is
$$\frac{\Gamma(n+1)\,\Gamma(1-m)}{\Gamma(n-m+2)}.$$

**Sol.** We have

$$\frac{\Gamma(n+1)\,\Gamma(1-m)}{\Gamma(n-m+2)}=B(n+1,\,1-m)$$

$$=\int_0^1 x^n(1-x)^{-m}\,dx$$

$$=\int_0^1 x^n\left(1+mx\frac{m(m+1)}{2!}x^2\right.$$
$$\left.+\frac{m(m+1)(m+2)}{3!}x^3+\ldots\right)dx$$

$$=\frac{1}{n+1}+m.\frac{1}{n+2}+\frac{m(m+1)}{2!}.\frac{1}{n+3}$$
$$+\frac{m(m+1)(m+2)}{2!}.\frac{1}{n+4}+\ldots$$

**Ex. 25.** *With certain restrictions on the values of a, b, m and n prove that*

$$\int_0^\infty\int_0^\infty \exp\{-(ax^2+by^2)\}.x^{2m-1}\,y^{2n-1}\,dx\,dy=\frac{\Gamma(m)\Gamma(n)}{4a^m b^m}.$$

**Sol.** Let $I=\int_0^\infty\int_0^\infty \exp\{-(ax^2+by^2)\}\,x^{2m-1}\,y^{2n-1}\,dx\,dy$

$$=\int_0^\infty e^{-ax^2}x^{2m-1}\,dx\times\int_0^\infty e^{-ay^2}y^{2n-1}\,dy$$

$$=I_1\times I_2.$$

To evaluate $I_1$ put $ax^2=t$, $2ax\,dx=dt$

$$\therefore\ I=\int_0^\infty e^{-t}\left(\frac{t}{a}\right)^{(2m-1/2}.\frac{dt}{2\sqrt{(at)}}$$

$$=\frac{1}{2a^m}\int_0^\infty e^{-t}.t^{m-1}dt=\frac{\Gamma(m)}{2a^m},$$

provided $a$ and $m$ are +ve.

Similarly $I_2=\frac{\Gamma(n)}{2b^n}$, provided $b$ and $n$ are +ve.

Hence $I=\frac{\Gamma(m)\,\Gamma(n)}{4a^m b^n}$

## Exercise on Chapter II

1. Evaluate

(i) $\int_0^\infty x^6 e^{2-x}\, dx$ (ii) $\int_0^\infty e^{-4x} x^{9/2}\, dx$

(ii) $\int_0^\infty 3^{-4x^2}\, dx.$

2. Show that

(i) $\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\, dt = \sqrt{(\pi/s)},\ s > 0$

(ii) $\int_0^1 x^{n-1}\left(\log \frac{1}{x}\right)^{m-1} dx = \frac{\Gamma(m)}{n^m},\ (m, n > 0),$

3. Show that

(i) $\int_0^\infty x^m e^{-x^n}\, dx = \frac{1}{n}\, \Gamma\left(\frac{m+1}{n}\right),\ (m > -1, n > 0)$

(ii) $\int_0^\infty e^{-ax} x^n\, dx = \frac{1}{a^{n+1}}\, \Gamma(n+1),\ (n > -1, a > 0).$

4. Show that

$$\int_0^\infty e^{(2ax-x^2)}\, dx = \tfrac{1}{2}\sqrt{\pi}\, e^{a^2}.$$

5. Prove that

$$\int_0^{\pi/2} \sqrt{(\tan \theta)}\, d\theta = \tfrac{1}{2}\, \Gamma(\tfrac{1}{4})\, \Gamma(\tfrac{3}{4}).$$

6. Evaluate

(i) $\int_{-1}^1 \left(\frac{1+x}{1-x}\right)^{1/2} dx$ (ii) $\int_0^\infty x^2 e^{-x^4}\, dx.\ \int_0^\infty e^{-x^4}\, dx.$

7. Prove that

$$\int_0^1 \frac{35x^3\, dx}{32\sqrt{(1-x)}} = 1.$$

8. Prove that

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin \theta)}} \times \int_0^{\pi/2} \sqrt{(\sin \theta)}\, d\theta = \pi$$ [Agra 80, 83]

9. Show that $3^{1/2}\, [\Gamma(\tfrac{1}{3})]^2 = \pi^{1/2}\, 2^{1/2}\, \Gamma(\tfrac{1}{6})$.

10. Find the value of

$$\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right)$$ $\left[\text{Ans. } \frac{16}{3}\pi^4\right]$

**11.** Show that

$$\Gamma(\cdot 1)\,\Gamma(\cdot 2)\,\Gamma(\cdot 3)\ldots\Gamma(\cdot 9)=\frac{(2\pi)^{9/2}}{\sqrt{(10)}}.$$

**12.** By means of the integral

$$\int_0^1 x^{m-1}\,(1-x^a)^n\,dx.$$

prove that

$$\frac{1}{mn\,!}-\frac{1}{(m+a)\,(n-1)\,!\,.\,1\,!}+\frac{1}{(m+2a).(n-2)\,!\,2\,!}+\ldots$$
$$+\frac{(-1)^n}{(m+na).n\,!}=\frac{a^n}{m(m+a)\,(m+2a)\ldots(m+na)}$$

Show that this integral may be expressed as

$$\frac{n\,!\,\Gamma\left(\frac{m}{a}\right)}{\left(\frac{m}{a}+n+1\right)}.$$

**13.** Prove that

$$\int_0^1 (1-x^n)^{1/n}\,dx=\frac{1}{n}\,\frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{2\Gamma\left(\frac{2}{n}\right)}.$$

**14.** Show that the perimeter of a loop of the curve

$$r^n=a^n\cos n\theta \text{ is } \frac{a}{n}.2^{(1/n)-1}.\frac{\left[\Gamma\left(\frac{1}{2n}\right)\right]^2}{\Gamma\left(\frac{1}{n}\right)}$$

**15.** Show that

$$\int_0^1 \frac{dx}{\sqrt{(1-x^n)}}=\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}\cdot\frac{\sqrt{\pi}}{n}$$

[Agra 72; Meerut 83 (P)]

**16.** Prove that

$$\int_0^\infty e^{-x^2}\,x^{2n}\,dx=\sqrt{\pi}\,\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\int_0^\infty e^{-x^2}\,x^{2n+1}\,dx$$

where $n$ is a positive integer.

**17.** Prove that

$$B(n,n)=\frac{\sqrt{\pi}\,\Gamma(n)}{2^{n-1}\,\Gamma(n+\frac{1}{2})}.$$

[Meerut 80 (S)]

18. Show that

$$B(n, n+1) = \frac{1}{2} \frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

and hence deduce that

$$\int_0^{\pi/2} \left( \frac{1}{\sin^3 \theta} - \frac{1}{\sin^2 \theta} \right)^{1/2} \cos \theta \, d\theta = \frac{\{\Gamma(\frac{1}{4})\}^2}{2\sqrt{\pi}}.$$

19. Prove that

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(b+cx)^{m+n}} dx = \frac{1}{(b+c)^m b^n} B(m, n).$$

[Meerut 83]

## Answers

1. (i) 45/8 (ii) $\frac{3}{128}\sqrt{\pi}$ (iii) $\frac{\sqrt{\pi}}{4\sqrt{(\log 3)}}$ (iv) $\frac{1}{n}\Gamma\left(\frac{m+1}{n}\right)$.

6. (i) $\pi$ (ii) $\frac{\pi}{16}\sqrt{2}$.

# The Dirac Delta Function

**§ 3·1. The Dirac delta function.** In mechanics we come across a very large force acting for only a very short interval of time, known as **impulsive force.** Thus we have functions which have non-zero values in very short intervals. The **Dirac delta function** may be thought of as a generalisation of this concept. The **Dirac delta function and its derivatives play a very useful role** in the solution of the boundary value problems in mathematical physics as well as in quantum mechanics.

Consider the function

$$\delta_\epsilon(t)=\begin{cases}\dfrac{1}{2\epsilon}, & |t|>\epsilon\\ 0, & |t|<\epsilon\end{cases} \qquad \ldots(1)$$

Thus $\displaystyle\int_{-\infty}^{\infty}\delta_\epsilon(t)\,dt=\int_{-\epsilon}^{\epsilon}\frac{1}{2\epsilon}\,dt$

$$=1. \qquad \ldots(2)$$

Again if $f(t)$ is any function which is integrable in the interval $(-\epsilon, \epsilon)$ then using the mean value theorem of the integral calculus, we have

$$\int_{-\infty}^{\infty} f(t)\,\delta_\epsilon(t)\,dt=\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} f(t)\,dt$$

$$=f(\theta_\epsilon),\quad |\theta|\leqslant 1. \qquad \ldots(3)$$

Thus we may think of a limiting function denoted by $\delta(t)$ approached by $\delta_\epsilon(t)$ as $\epsilon\to 0$.

*i.e.*

$$\delta(t)=\lim_{\epsilon\to 0}\delta_\epsilon(t) \qquad \ldots(4)$$

As $\epsilon\to 0$, from (1) and (2), we have

$$\delta(t)=\lim_{\epsilon\to 0}\delta_\epsilon(t)=\begin{cases}\infty \text{ if } t=0\\ 0 \text{ if } t\neq 0\end{cases} \qquad \ldots(5)$$

and

$$\int_{-\infty}^{\infty}\delta(t)\,dt=1. \qquad \ldots(6)$$

This limiting function $\delta(t)$ defined by equations (5) and (6) is known as the **Dirac delta function** (or the unit impulse function)

after Dirac who first introduced it. Dirac called this delta fnnction as improper function as there can not be proper function satisfying such conditions.

**§ 3·2. Delta function $\delta\,(x-a)$. (Defnition).**

We define $\delta(x-a)=\begin{cases}\infty & \text{if } x=a\\ 0 & \text{if } x\neq a\end{cases}$

together with the condition

$$\int_{-\infty}^{\infty}\delta\,(x-a)\,dx=1$$

**§ 3·3. Properties of the Dirac delta function.**

**Prop.** $$\int_{-\infty}^{\infty}\delta(t)\ dt=1.$$

**Prop. 2.** $$\int_{-\infty}^{\infty} f(t)\ \delta(t)\ dt=f(0)$$

for any continuous function $f(x)$.

**Proof.** Taking the limit as $\epsilon\to 0$ in equatien (3), we have

$$\int_{-\infty}^{\infty} f(t)\ dt=f(0).$$

**Prop. 3.** $$\int \delta\,(t-a)\ f(t)\ dt=f(a)$$

for any continuous function $f(t)$.

**Proof.** Consider the function

$$\delta_\epsilon(t-a)=\begin{cases}\dfrac{1}{\epsilon}, & a<t<a+\epsilon\\ 0 & \text{elsewhere}\end{cases}$$

Then using the mean value theorem of the integral calculus

$$\int_{-\infty}^{\infty}\delta_\epsilon(t-a).f(t)\ dt=\frac{1}{\epsilon}\int_a^{a+\epsilon} f(t)\ dt$$

$$=\frac{1}{\epsilon}(a+\epsilon-\theta)\,f\,(a+\theta\pi)$$

where $0<\theta<1$

$$=f\,(a+\theta\epsilon)$$

on taking the limit as $\epsilon\to 0$, we have

$$\int_{-\infty}^{\infty}\delta\,(t-a)f(t)\ dt=f(a).$$

**Thus the operation of multiplying f(t) by $\delta$ (t$-a$) and integra-ing over all x is merely equivalent to substituting $a$ for x in the original function f(t).**

Symbolically we may write

$$f(t)\ \delta\ (t-a)=f(a)\ \delta\ (t-a).$$

As a speacial case we have

$$x,\ \delta(x)=0.$$

**Prop. 4.** $\delta(-x)=\delta\ (x)$.

**Proof.** $\delta(-x)$ satisfies the relations.

$$\delta\ (-x)=\begin{cases}0 & \text{if} x\neq 0\\ \infty & \text{if } x=0\end{cases}$$

and $$\int_{-\infty}^{\infty} \delta\ (-x)\ dx=-\int_{\infty}^{-\infty} \delta\ (t)\ dt$$

Putting $x=-t$ so that $dx=-dt$

$$=\int_{-\infty}^{\infty} \delta(x)\ dx=1.$$

Hence $\delta(-x)=\delta(x)$.

**Prop. 5.** $\delta(\mathbf{ax})=\frac{1}{\mathbf{a}}\delta\ (\mathbf{x}),\ \mathbf{a}>0$.

**Proof.** Since $x\neq 0 \Leftrightarrow ax\neq 0\ \because a>0$

$\therefore\ \delta\ (ax)$ satisfies the relation

$$\delta\ (ax)=\int_{\infty}^{0} \begin{matrix}\text{if } ax\neq 0\\ \text{if } ax=0\end{matrix}$$

and $$\int_{-\infty}^{\infty} a\delta\ (ax)\ dx=\int_{-\infty}^{\infty} \delta\ (t)\ dt$$

Putting $ax=t$ so that $dx=\frac{1}{a}\ dt$

$$=\int_{-\infty}^{\infty} \delta(x)\ dx=1$$

which implies that

$$a\delta\ (ax)=\delta(x)$$

or $$\delta(ax)=\frac{1}{a}\delta(x).$$

## § 3·4. Derivatives of δ (x).

**If $\delta\ (x)$ is continuous differentiable 'Dirac delta Function, vanishing for large x, then**

$$\int_{-\infty}^{\infty} f(x)\ \delta'\ (x)\ dx=-f'\ (0).$$

**Proof.** Assuming that $f(x)$, $\delta\ (x)$ and $\delta'\ (x)$ can be regarded as ordinary functions then by the rule of integrating by parts, we have

$$\int_{-\infty}^{\infty} f(x)\, \delta'(x)\, dx = \Big[ f(x)\, \delta(x) \Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\, \delta(x)\, dx$$

$$= -f'(0) \text{ By Prop. 3.}$$

**Note.** Repeating this process again and again, we have

$$\int_{-\infty}^{\infty} f(x)\, \delta^n(x)\, dx = (-1)^n f^n(0).$$

**3·5. The Heaviside Unit Function (or the Unit Step Function).**

The heaviside unit step function denoted by $H(x)$ is defined as

$$H(x) = \begin{cases} 1 & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$$

If $a > 0$, then

$$H(x-a) = \begin{cases} 1 & \text{if } x \geqslant a \\ 0 & \text{if } x < a \end{cases}$$

**§ 3·6. The Dirac delta function is the derivative of the Heaviside unit function H(x) defined by the equation**

$$H(x) = \begin{cases} 1 & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$$

**Proof.** We have

$$\int_{-\infty}^{\infty} f(x)\, H'(x)\, dx = \lim_{a\to\infty} \int_{-a}^{a} f(x)\, H'(x)\, dx.$$

Integrating by parts taking $H'(x)$ as second function

$$= \lim_{a\to\infty} \left[ \Big\{ f(x)\, H(x) \Big\}_{-a}^{a} - \int_{-a}^{a} f'(x)\, H(x)\, dx \right]$$

$$= \lim_{a\to\infty} [f(a)\, H(a) - f(-a)\, H(-a)]$$

$$- \int_{-a}^{0} f'(x)\, H(x)\, dx - \int_{0}^{a} f'(x)\, H(x)\, dx$$

$$= \lim_{a\to\infty} \left[ f(a) - 0 - \int_{0}^{a} f'(x)\, dx \right]$$

$$= \lim_{a\to\infty} \left[ f(a) - \Big\{ f(x) \Big\}_{0}^{a} \right]$$

$$= \lim_{a\to\infty} [f(a) - f(a) + f(0)]$$

$$= f(0)$$

$$= \int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx \quad \text{from Prop. 2.}$$

which implies that $H'(x) = \delta(x)$.

## Exercise on Chapter III

1. Show that

$$\delta(a^2-x^2)=\frac{1}{2a}[\delta(x-a)+\delta(x+a)],\ a>0.$$

2. For the derivative $\delta'(x)$ of $\delta(x)$ prove that

(i) $\delta'(x)=-\delta'(-x)$

(ii) $x\delta'(x)=-\delta(x)$

(iii) $x^2\,\delta'(x)=0$

and (iv) $\int \delta'(x-y)\,\delta(y-a)\,dy=\delta'(x-a)$.

3. Show that

$$\int_0^\infty (\cos xy)(\cos xy')\,dx=\tfrac{1}{2}\pi\,(y-y').$$

4. Show that the function.

$$\lim_{\epsilon\to 0} \sin(2\pi\,\epsilon x)/\pi x.$$

is a Dirac delta function.